$\boxed{1}$

(1)

解と係数の関係より $\alpha+\beta=3, \alpha\beta=1$.

$$\begin{aligned} t_1 &= \alpha+\beta=3 \\ t_2 &= \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=7 \end{aligned}$$

(2)

$$\alpha^{n+2}+\beta^{n+2}=(\alpha+\beta)(\alpha^{n+1}+\beta^{n+1})-\alpha\beta(\alpha^n+\beta^n)$$

よって, $A=\alpha+\beta=3, B=-\alpha\beta=-1$.

(3)

$$\begin{aligned} t_7 &= 3t_6-t_5=8t_5-3t_4=21t_4-8t_3=55t_3-21t_2 \\ &= 144t_2-55t_1=843 \end{aligned}$$

$\boxed{2}$

(1)

$\angle\mathrm{AOB} = \theta$ とすると, 余弦定理より

$$\cos\theta = \frac{\mathrm{OA}^2 + \mathrm{OB}^2 - \mathrm{AB}^2}{2 \cdot \mathrm{OA} \cdot \mathrm{OB}} = -\frac{5}{6}$$

よって,

$$\overrightarrow{a} \cdot \overrightarrow{b} = \mathrm{OA} \cdot \mathrm{OB}\cos\theta = -10$$

(2)

$$\overrightarrow{\mathrm{OE}} = \frac{1}{1+t}\overrightarrow{\mathrm{OC}} + \frac{t}{1+t}\overrightarrow{\mathrm{OB}} = \frac{1}{2(1+t)}\overrightarrow{a} + \frac{t}{1+t}\overrightarrow{b}$$
$$\overrightarrow{\mathrm{OF}} = \frac{1}{1+t}\overrightarrow{\mathrm{OA}} + \frac{t}{1+t}\overrightarrow{\mathrm{OD}} = \frac{1}{1+t}\overrightarrow{a} + \frac{t}{2(1+t)}\overrightarrow{b}$$

(3)

$$\overrightarrow{\mathrm{OE}} \cdot \overrightarrow{\mathrm{OF}} = \frac{1}{2(1+t)^2}|\overrightarrow{a}|^2 + \frac{5t}{4(1+t)^2}\overrightarrow{a} \cdot \overrightarrow{b} + \frac{t^2}{2(1+t)^2}|\overrightarrow{b}|^2$$
$$= \frac{9t^2 - 25t + 16}{2(1+t)^2}$$

$\overrightarrow{\mathrm{OE}}$ と $\overrightarrow{\mathrm{OF}}$ が直交するとき $\overrightarrow{\mathrm{OE}} \cdot \overrightarrow{\mathrm{OF}} = 0$ より,

$$9t^2 - 25t + 16 = 0 \quad \therefore t = 1, \frac{16}{9}$$

$\boxed{3}$

(1)

余弦定理より,

$$
\begin{aligned}
\mathrm{AB}^2 &= \mathrm{OA}^2 + \mathrm{OB}^2 - 2\mathrm{OA}\cdot\mathrm{OB}\cos\angle\mathrm{AOB} \\
&= 1 + 4 - 4\cos 60^\circ = 3 \quad \therefore \mathrm{AB} = \sqrt{3} \\
\mathrm{BC}^2 &= \mathrm{OB}^2 + \mathrm{OC}^2 - 2\mathrm{OB}\cdot\mathrm{OC}\cos\angle\mathrm{BOC} \\
&= 4 + 2 - 4\sqrt{2}\cos 45^\circ = 2 \quad \therefore \mathrm{BC} = \sqrt{2} \\
\mathrm{CA}^2 &= \mathrm{OC}^2 + \mathrm{OA}^2 - 2\mathrm{OC}\cdot\mathrm{OA}\cos\angle\mathrm{COA} \\
&= 2 + 1 - 2\sqrt{2}\cos 45^\circ = 1 \quad \therefore \mathrm{CA} = 1
\end{aligned}
$$

(2)

余弦定理より

$$\cos\angle\mathrm{BCA} = \frac{\mathrm{BC}^2 + \mathrm{CA}^2 - \mathrm{AB}^2}{2\mathrm{BC}\cdot\mathrm{CA}} = 0$$

$0^\circ < \angle\mathrm{BCA} < 180^\circ$ より, $\angle\mathrm{BCA} = 90^\circ$.

(3)

$$
\begin{aligned}
\cos\angle\mathrm{OAB} &= \frac{\mathrm{OA}^2 + \mathrm{AB}^2 - \mathrm{OB}^2}{2\mathrm{OA}\cdot\mathrm{AB}} = 0 \\
\cos\angle\mathrm{OAC} &= \frac{\mathrm{OA}^2 + \mathrm{CA}^2 - \mathrm{OC}^2}{2\mathrm{OA}\cdot\mathrm{CA}} = 0
\end{aligned}
$$

よって, $\mathrm{OA}\perp\mathrm{AB}$ かつ $\mathrm{OA}\perp\mathrm{AC}$ より OA は $\triangle\mathrm{ABC}$ に垂直である. ここで, $\triangle\mathrm{ABC}$ の面積は $\angle\mathrm{BCA} = 90^\circ$ より,

$$\frac{1}{2}\mathrm{BC}\cdot\mathrm{CA} = \frac{\sqrt{2}}{2}$$

よって, 求める四面体 OABC の体積は

$$V = \frac{1}{3}\cdot\frac{\sqrt{2}}{2}\cdot 1 = \frac{\sqrt{2}}{6}$$

(4)

$\triangle\mathrm{OBC}$ の面積は

$$\frac{1}{2}\mathrm{OB}\cdot\mathrm{OC}\sin\angle\mathrm{BOC} = 1$$

よって, $V = \dfrac{1}{3}\cdot 1\cdot\mathrm{AH} = \dfrac{\sqrt{2}}{6}$ より $\mathrm{AH} = \dfrac{\sqrt{2}}{2}$.

$\boxed{4}$

(A)

(1)

$$\int_0^{\log 3} \frac{e^x}{e^x+1}dx = \left[\log(e^x+1)\right]_0^{\log 3} = \log 2$$

(2)

$$(f(x))^2 = \frac{e^{2x}}{(e^x+1)^2}$$
$$f(x) - f'(x) = \frac{e^x}{e^x+1} - \frac{e^x}{(e^x+1)^2} = \frac{e^{2x}}{(e^x+1)^2}$$

(3)

$$\begin{aligned} V &= \pi \int_0^{\log 3} (f(x))^2\,dx = \pi \int_0^{\log 3} (f(x) - f'(x))\,dx \\ &= \pi \left[\log(e^x+1) - \frac{e^x}{e^x+1}\right]_0^{\log 3} \\ &= \left(\log 2 - \frac{1}{4}\right)\pi \end{aligned}$$

(B)

(1)

$$\frac{1}{3}(x-1)f'(x) - 4x + 4 = x^3 + \frac{2a-3}{3}x^2 - \frac{2a-b+12}{3}x - \frac{b}{3} + 4$$

$f(x)$ と係数を比較して,

$$a = \frac{2a-3}{3}, b = -\frac{2a-b+12}{3}, c = -\frac{b}{3} + 4$$
$$\Leftrightarrow a = -3, b = -3, c = 5$$

(2)

$$\begin{aligned} \int_{-1}^{1} |f'(x)| &= \int_{-1}^{1} \left|3x^2 - 6x - 3\right|dx \\ &= \int_{-1}^{-\sqrt{2}+1} (3x^2 - 6x - 3)dx + \int_{-\sqrt{2}+1}^{1} (-3x^2 + 6x + 3)dx \\ &= 8\sqrt{2} - 4 \end{aligned}$$