$\boxed{\text{I}}$

(1)

$$a_{n+2}=\frac{1}{6}(a_{n+1}+a_n+8)$$

$$(a_n=b_n+\alpha \quad \alpha>0)$$

$$\iff b_{n+2}+\alpha=\frac{1}{6}(b_{n+1}+\alpha+b_n+\alpha+8)$$

$$\iff b_{n+2}=\frac{1}{6}(b_{n+1}+b_n)+\underbrace{\frac{\alpha}{3}+\frac{4}{3}-\alpha}_{0} \qquad \therefore\ \alpha=\overset{\text{ア}}{\boxed{2}} \quad (\text{答})$$

このとき，

$$b_{n+2}=\frac{1}{6}(b_{n+1}+b_n)$$

$$\iff \begin{cases} b_{n+2}+\frac{1}{3}b_{n+1}=\frac{1}{2}\left(b_{n+1}+\frac{1}{3}b_n\right) \leftarrow c_n=b_{n+1}+\frac{1}{3}b_n & \therefore\ \beta=\dfrac{\overset{\text{イ}}{\boxed{1}}}{\underset{\text{ウ}}{\boxed{3}}} \quad (\text{答}) \\ b_{n+2}-\frac{1}{2}b_{n+1}=-\frac{1}{3}\left(b_{n+1}-\frac{1}{2}b_n\right) \leftarrow d_n=b_{n+1}-\frac{1}{2}b_n & \therefore\ \gamma=\dfrac{\overset{\text{エ}}{\boxed{1}}}{\underset{\text{オ}}{\boxed{2}}} \quad (\text{答}) \end{cases}$$

$$\iff \begin{cases} c_{n+1}=\frac{1}{2}c_n & (c_1=3) \\ d_{n+1}=-\frac{1}{3}c_n & (d_1=2) \end{cases}$$

よって，

$$c_n=3\cdot\left(\frac{1}{2}\right)^{n-1}=\gamma^{n-1}\cdot\overset{\text{カ}}{\boxed{3}} \quad (\text{答})$$

$$d_n=2\cdot\left(-\frac{1}{3}\right)^{n-1}=(-\beta)^{n-1}\cdot\overset{\text{キ}}{\boxed{2}} \quad (\text{答})$$

以上より，

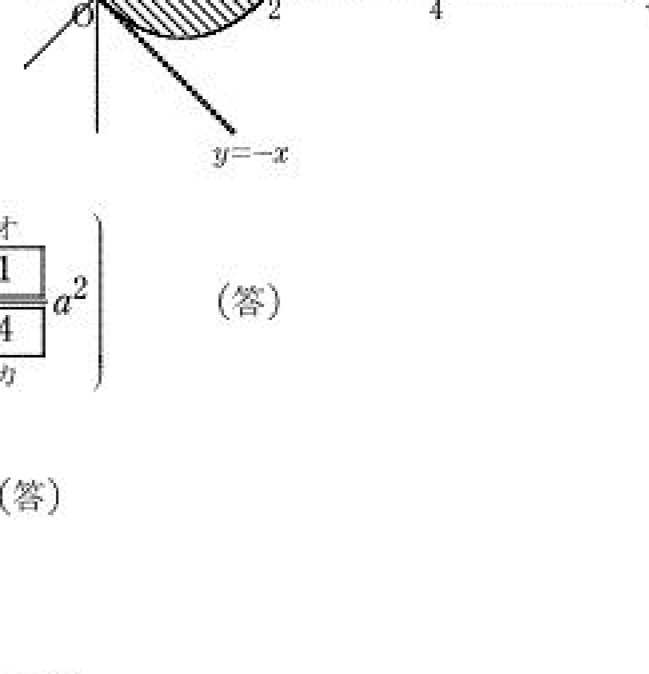
$$\begin{cases} b_{n+1}+\frac{1}{3}b_n=\gamma^{n-1}\cdot 3 \\ b_{n+1}-\frac{1}{2}b_n=(-\beta)^{n-1}\cdot 2 \end{cases}$$

$$\therefore\ b_n=\frac{6}{5}\left\{\gamma^{n-1}\cdot 3-(-\beta)^{n-1}\cdot 2\right\}$$

$$a_n=b_n+2=b_n+\alpha$$

$$\therefore\ a_n=\gamma^{n-1}\cdot\frac{\overset{\text{クケ}}{\boxed{18}}}{\underset{\text{コ}}{\boxed{5}}}-(-\beta)^{n-1}\cdot\frac{\overset{\text{サシ}}{\boxed{12}}}{\underset{\text{ス}}{\boxed{5}}}+\alpha \quad (\text{答})$$

(2)



$$\frac{x^2}{2}-x=x \iff x^2-4x=0$$

$$\iff x(x-4)=0 \qquad \therefore\ x=0,4$$

$$\therefore\ \text{A}\left(\overset{\text{ア}}{\boxed{4}},\overset{\text{ア}}{\boxed{4}}\right) \quad (\text{答})$$

$$(\text{Rの面積})=-\frac{1}{2}\int_0^4 x(x-4)dx=\frac{4^3}{12}=\frac{\overset{\text{イウ}}{\boxed{16}}}{\underset{\text{エ}}{\boxed{3}}} \quad (\text{答})$$

$$\begin{cases} y=-(x-a)+\frac{a^2}{2}-a \\ y=x \end{cases} \qquad \therefore\ x=\frac{a^2}{4}$$

$$\therefore\ \text{Q}\left(\frac{\overset{\text{オ}}{\boxed{1}}}{\underset{\text{カ}}{\boxed{4}}}a^2,\frac{\overset{\text{オ}}{\boxed{1}}}{\underset{\text{カ}}{\boxed{4}}}a^2\right) \quad (\text{答})$$

$$\therefore\ \overline{\text{PQ}}=\sqrt{2}\left|\underset{(+)}{a-\frac{1}{4}a^2}\right|=\sqrt{\overset{\text{キ}}{\boxed{2}}}\,a\left(1-\frac{\overset{\text{ク}}{\boxed{1}}}{\underset{\text{ケ}}{\boxed{4}}}a\right) \quad (\text{答})$$

$\overline{\text{OQ}}=l$ とすると，

$$l=\frac{\sqrt{2}}{4}a^2 \qquad \therefore\ dl=\frac{\sqrt{2}}{2}ada$$

l	$0 \to 4\sqrt{2}$
a	$0 \to 4$

$$V=\pi\int_0^{4\sqrt{2}}\overline{\text{PQ}}^2dl$$

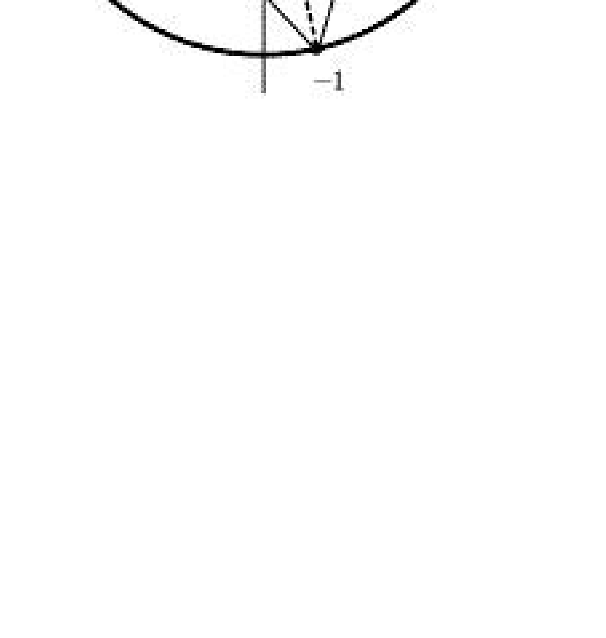
$$=2\pi\int_0^4 a^2\left(1-\frac{a}{4}\right)^2\cdot\frac{\sqrt{2}}{2}ada$$

$$=\sqrt{2}\pi\int_0^4\left(a^3-\frac{a^4}{2}+\frac{a^5}{16}\right)da$$

$$=\sqrt{2}\pi\left[\frac{a^4}{4}-\frac{a^5}{10}+\frac{a^6}{96}\right]_0^4$$

$$=\frac{\overset{\text{コサ}}{\boxed{64}}\sqrt{\overset{\text{シ}}{\boxed{2}}}}{\underset{\text{スセ}}{\boxed{15}}}\pi \quad (\text{答})$$

(3)



条件より

$$z_1=\cos\left(\pm\frac{2\pi}{3}\right)+i\sin\left(\pm\frac{2\pi}{3}\right)$$

$$=\frac{\overset{\text{アイ}}{\boxed{-1}}}{\underset{\text{ウ}}{\boxed{2}}}\pm\frac{\sqrt{\overset{\text{エ}}{\boxed{3}}}}{\underset{\text{オ}}{\boxed{2}}}i \quad (\text{答})$$

次に， $z_2=-\sqrt{3}-1+ai$ （a：実数）に対し，

$$\overline{z_2}=-\sqrt{3}-1-ai$$

$$z=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$$

ここで，

$$z\overline{z_2}-z=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)(zz_2-z)$$

$$\iff \overline{z_2}-1=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(z_2-1)$$

$$\iff -\sqrt{3}-2-ai=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(-\sqrt{3}-2+ai)$$

$$\iff -\sqrt{3}-2-ai=-\frac{\sqrt{3}}{2}-1-\frac{\sqrt{3}}{2}a+\left(\frac{a}{2}-\frac{3}{2}-\sqrt{3}\right)i$$

よって，

$$\begin{cases} -\sqrt{3}-2=-\frac{\sqrt{3}}{2}-1-\frac{\sqrt{3}}{2}a \\ a=-\frac{a}{2}+\frac{3}{2}+\sqrt{3} \end{cases} \qquad \therefore\ a=1+\frac{2}{3}\sqrt{3}$$

$z\overline{z_2}-z=\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}(zz_2-z)$ も同様に考えれば，

$$\therefore\ a=\pm\frac{3+2\sqrt{3}}{3}$$

$$\therefore\ z_2=-\sqrt{3}-1\pm\frac{\overset{\text{カ}}{\boxed{3}}+\overset{\text{キ}}{\boxed{2}}\sqrt{\overset{\text{ク}}{\boxed{3}}}}{\underset{\text{ケ}}{\boxed{3}}}i \quad (\text{答})$$

次に，

$$|z_3|^2=|z_2|^2=(\sqrt{3}+1)^2+\left(1+\frac{2}{3}\sqrt{3}\right)^2$$

$$=4+2\sqrt{3}+1+\frac{4}{3}+\frac{4}{3}\sqrt{3}$$

$$=\frac{19}{3}+\frac{10}{3}\sqrt{3}$$

$$z_3=p+qi \ (p,q:\text{実数}) \qquad (\overline{z_3}=p-qi)$$

$$p^2+q^2=\frac{1}{3}(19+10\sqrt{3}) \ \cdots\cdots①$$

$$z\overline{z_3}-z=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)(zz_3-z)$$

$$\iff \overline{z_3}-1=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(z_3-1)$$

$$\iff p-1-qi=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(p-1+qi)$$

$$\iff p-1-qi=\frac{p}{2}-\frac{1}{2}-\frac{\sqrt{3}}{2}q+\left(\frac{\sqrt{3}}{2}p-\frac{\sqrt{3}}{2}+\frac{q}{2}\right)i$$

よって， $\begin{cases} p-1=\frac{p}{2}-\frac{1}{2}-\frac{\sqrt{3}}{2}q \\ q=-\frac{\sqrt{3}}{2}p+\frac{\sqrt{3}}{2}-\frac{q}{2} \end{cases} \qquad \therefore\ p+\sqrt{3}q=1 \ \cdots\cdots②$

①，②より，$p>0$ に注意すると， $p=\dfrac{3+2\sqrt{3}}{2}$

よって，

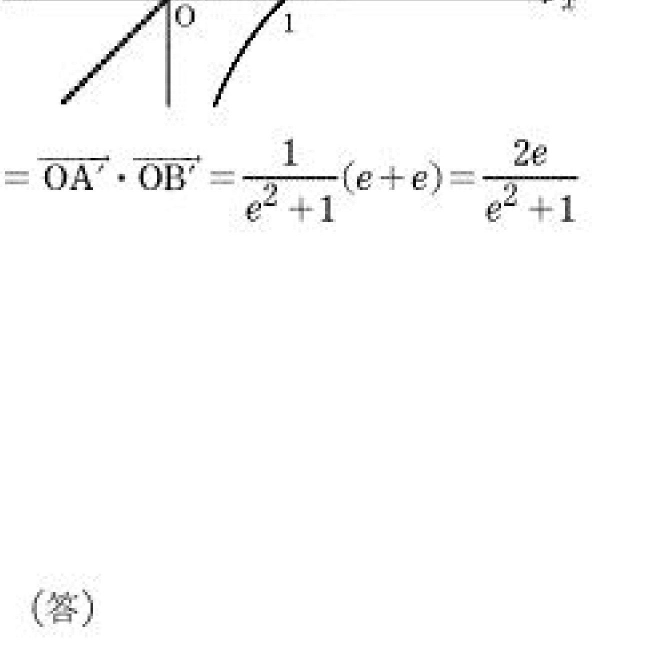
$$q=\frac{1}{\sqrt{3}}\left(1-\frac{3+2\sqrt{3}}{2}\right)=-1-\frac{\sqrt{3}}{6}=-\frac{6+\sqrt{3}}{6}$$

$z\overline{z_3}-z=\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}(zz_3-z)$ も同様に考えれば，

$$\therefore\ q=\frac{6+\sqrt{3}}{6}$$

$$\therefore\ z_3=\frac{\overset{\text{コ}}{\boxed{3}}+\overset{\text{サ}}{\boxed{2}}\sqrt{\overset{\text{シ}}{\boxed{3}}}}{\underset{\text{ス}}{\boxed{2}}}\pm\frac{\overset{\text{セ}}{\boxed{6}}+\sqrt{\overset{\text{ソ}}{\boxed{3}}}}{\underset{\text{タ}}{\boxed{6}}}i \quad (\text{答})$$

(4) $\overrightarrow{\text{OA}'}$ と $\overrightarrow{\text{OB}'}$ の内積が最大となるのは，
直線OAが $y=\log x$ に接するときの接点がA
（同様に， 直線OBが $y=e^x$ に接するときの接点がB）
となるとき，このとき， AとBは $y=x$ に対称．



$\text{A}(a,\log a)$ とおくと，直線OAの方程式は

$$y=\frac{1}{a}(x-a)+\log a$$

$$\iff y=\frac{1}{a}x+\underbrace{\log a-1}_{0} \qquad \therefore\ a=e$$

$$\therefore\ \overrightarrow{\text{OA}}=(e,1),\ \overrightarrow{\text{OA}'}=\frac{1}{\sqrt{e^2+1}}(e,1)$$

$$\therefore\ \overrightarrow{\text{OB}}=(1,e),\ \overrightarrow{\text{OB}'}=\frac{1}{\sqrt{e^2+1}}(1,e)$$

よって，$m=\overrightarrow{\text{OA}'}\cdot\overrightarrow{\text{OB}'}=\dfrac{1}{e^2+1}(e+e)=\dfrac{2e}{e^2+1}$

$$\therefore\ \log m=\log\frac{2e}{e^2+1}$$

$$=\log 2+\log e-\log(e^2+1)$$

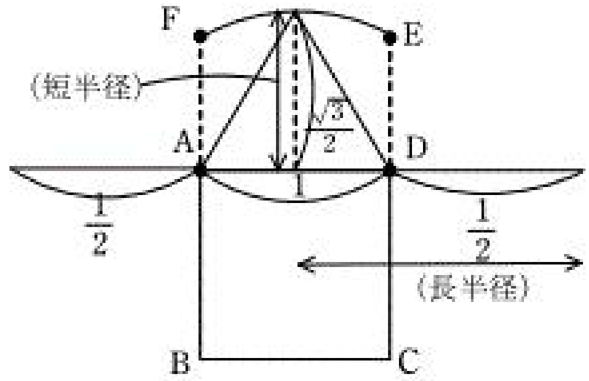
$$=\overset{\text{ア}}{\boxed{1}}+\log\overset{\text{イ}}{\boxed{2}}-\log\left(e^{\overset{\text{ウ}}{\boxed{2}}}+\overset{\text{エ}}{\boxed{1}}\right) \quad (\text{答})$$

Ⅱ

$$\left(\begin{array}{l}\text{問題文図 I の}\\\text{斜線部分の面積}\end{array}\right)=\left(\frac{1}{2}\cdot 5^2\cdot\alpha\right)\cdot\frac{3}{5}=\frac{\overset{\text{アイ}}{\boxed{15}}}{\underset{\text{ウ}}{\boxed{2}}}\alpha \quad \text{(答)}$$

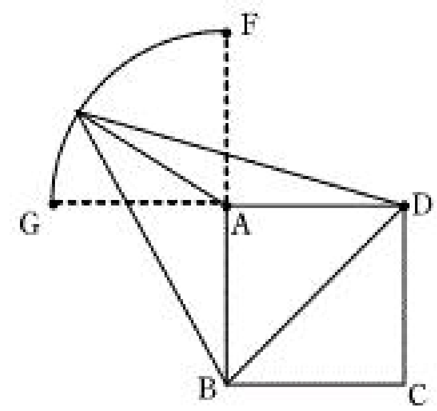
$$\text{(長半径)}=\frac{1}{2}+\frac{1}{2}=\overset{\text{エ}}{\boxed{1}} \quad \text{(答)}$$

$$\text{(短半径)}=\frac{\sqrt{\overset{\text{オ}}{\boxed{3}}}}{\underset{\text{カ}}{\boxed{2}}} \quad \text{(答)}$$



$$\text{(長半径)}=\frac{\overset{\text{キ}}{\boxed{3}}}{\underset{\text{ク}}{\boxed{2}}} \quad \text{(答)}$$

$$\text{(短半径)}=\frac{\sqrt{\overset{\text{ケ}}{\boxed{7}}}}{\underset{\text{コ}}{\boxed{2}}} \quad \text{(答)}$$



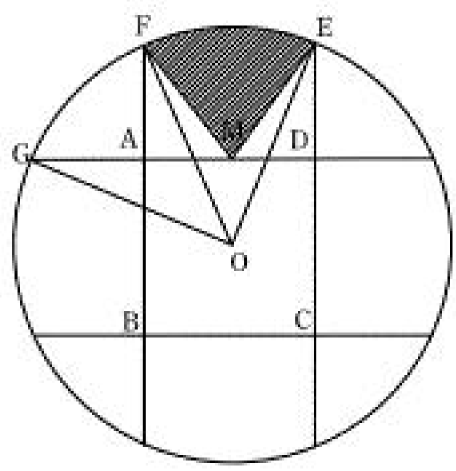
$$\text{(斜線部分の面積)}=\left(\frac{1}{2}\cdot 1^2\cdot\frac{\pi}{3}\right)\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{\overset{\text{サ}}{\boxed{3}}}}{\underset{\text{シス}}{\boxed{12}}}\pi \quad \text{(答)}$$

$$1+1+1+\mathrm{AF}+\sqrt{1+\mathrm{AF}^2}=5$$

$$\Longleftrightarrow\quad \sqrt{1+\mathrm{AF}^2}=2-\mathrm{AF}$$

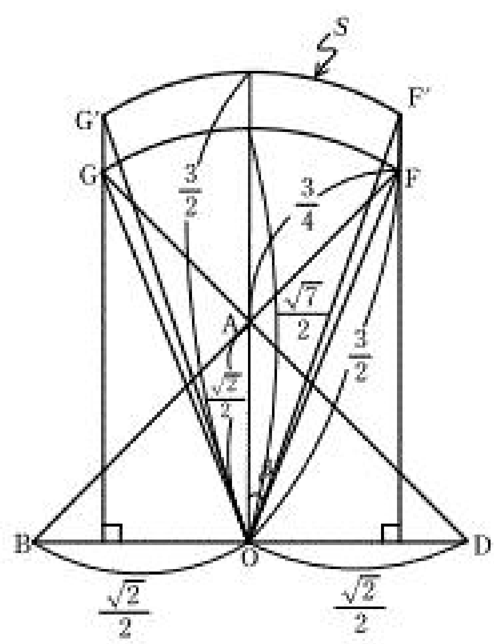
$$\therefore\ 1+\mathrm{AF}^2=4-4\mathrm{AF}+\mathrm{AF}^2$$

$$\therefore\ \mathrm{AF}=\frac{\overset{\text{セ}}{\boxed{3}}}{\underset{\text{ソ}}{\boxed{4}}} \quad \text{(答)}$$



$$\therefore\ \sin\beta=\frac{\mathrm{G'F'}}{\frac{3}{2}\cdot 2}=\frac{\mathrm{GF}}{3}=\frac{\sqrt{2}\mathrm{AF}}{3}=\frac{\frac{3}{4}\sqrt{2}}{3}=\frac{\sqrt{\overset{\text{タ}}{\boxed{2}}}}{\underset{\text{チ}}{\boxed{4}}} \quad \text{(答)}$$

$$\left\{\frac{1}{2}\cdot\left(\frac{3}{2}\right)^2\cdot 2\beta\right\}\cdot\frac{\frac{\sqrt{7}}{2}}{\frac{3}{2}}=\frac{9}{4}\cdot\frac{\sqrt{7}}{3}\beta=\frac{\overset{\text{ツ}}{\boxed{3}}\sqrt{\overset{\text{テ}}{\boxed{7}}}}{\underset{\text{ト}}{\boxed{4}}}\beta \quad \text{(答)}$$



以上より，軌跡によって囲まれる部分の面積は，

$$\left(\frac{\sqrt{3}}{12}\pi+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 2\right)\cdot 4+\frac{3\sqrt{7}}{4}\beta\cdot 4$$

$$=\frac{\sqrt{\overset{\text{ナ}}{\boxed{3}}}}{\underset{\text{ニ}}{\boxed{3}}}\pi+\overset{\text{ヌ}}{\boxed{3}}\sqrt{\overset{\text{ネ}}{\boxed{7}}}\beta+\overset{\text{ノ}}{\boxed{1}} \quad \text{(答)}$$

Ⅲ

(1)

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\\
&=-\left(\frac{a+b}{2}\right)^2-\frac{1}{2}\left(-a^2-b^2\right)\\
&=-\frac{1}{4}\left(a^2+2ab+b^2-2a^2-2b^2\right)\\
&=\frac{1}{4}\left(a^2-2ab+b^2\right)\\
&=\frac{1}{4}(a-b)^2\geqq 0 \qquad \therefore\ f\left(\frac{a+b}{2}\right)\geqq\frac{f(a)+f(b)}{2}
\end{aligned}
$$

（等号成立は $a=b$ のとき）（証明終）

(2)

「$f\left(\dfrac{a_1+a_2+a_3+\cdots\cdots+a_{2^k}}{2^k}\right)\geqq\dfrac{f(a_1)+f(a_2)+\cdots\cdots+f\left(a_{2^k}\right)}{2^k}$ $(k=1,2,3\cdots\cdots)$」$\cdots\cdots(*)$

を帰納法で示す．

（Ⅰ）$k=1$ のとき，$f\left(\dfrac{a_1+a_2}{2}\right)\geqq\dfrac{f(a_1)+f(a_2)}{2}$ となり，性質(A)より明らかに成立．

（Ⅱ）$k=i$ $(\geqq 1)$ のとき成立すると仮定する．つまり，

$$
f\left(\frac{a_1+a_2+a_3+\cdots\cdots+a_{2^i}}{2^i}\right)\geqq\frac{f(a_1)+f(a_2)+\cdots\cdots+f\left(a_{2^i}\right)}{2^i}\ \cdots\cdots①
$$

が成立するとする．ここで，

$$
\begin{aligned}
&f\left(\frac{a_1+a_2+\cdots\cdots+a_{2^{i+1}}}{2^{i+1}}\right)=f\left(\frac{a_1+a_2+\cdots\cdots+a_{2^i}+a_{2^i+1}+\cdots\cdots+a_{2^{i+1}}}{2\cdot 2^i}\right)\\
&=f\left(\frac{\dfrac{a_1+a_2+\cdots\cdots+a_{2^i}}{2^i}+\dfrac{a_{2^i+1}+\cdots\cdots+a_{2^{i+1}}}{2^i}}{2}\right)
\end{aligned}
$$

ともに 2^i 個ずつある

①を利用

$$
\begin{aligned}
&\geqq\frac{1}{2}\left\{f\left(\frac{a_1+a_2+\cdots\cdots+a_{2^i}}{2^i}\right)+f\left(\frac{a_{2^i+1}+\cdots\cdots+a_{2^{i+1}}}{2^i}\right)\right\}\\
&\geqq\frac{1}{2}\left\{\frac{f(a_1)+f(a_2)+\cdots\cdots+f\left(a_{2^i}\right)}{2^i}+\frac{f\left(a_{2^i+1}\right)+f\left(a_{2^i+2}\right)+\cdots\cdots+f\left(a_{2^{i+1}}\right)}{2^i}\right\}\\
&=\frac{f(a_1)+f(a_2)+\cdots\cdots+f\left(a_{2^i}\right)+\cdots\cdots+f\left(a_{2^{i+1}}\right)}{2^{i+1}}
\end{aligned}
$$

となり，$k=i+1$ のときでも成立．

以上(Ⅰ)，(Ⅱ)より，$(*)$は成立．　（証明終）

(3)　性質(A)をみたす関数 $y=f(x)$ の $0\leqq x\leqq 1$ の区間を 2^k 等分し，

小区間の幅を $\dfrac{1}{2^k}$ とすると，(2)より，

$$
\begin{aligned}
&\frac{f\left(\dfrac{1}{2^k}\right)+f\left(\dfrac{2}{2^k}\right)+\cdots\cdots+f\left(\dfrac{2^k}{2^k}\right)}{2^k}\leqq f\left(\frac{\dfrac{1}{2^k}+\dfrac{2}{2^k}+\cdots\cdots+\dfrac{2^k}{2^k}}{2^k}\right)\\
&\iff \frac{1}{2^k}\sum_{i=1}^{2^k}f\left(\frac{i}{2^k}\right)\leqq f\left(\frac{1}{2^{2k}}\cdot\frac{2^k\left(2^k+1\right)}{2}\right)\\
&\iff \frac{1}{2^k}\sum_{i=1}^{2^k}f\left(\frac{i}{2^k}\right)\leqq f\left(\frac{1}{2}\left(1+\frac{1}{2^k}\right)\right)
\end{aligned}
$$

ここで，辺々極限$(k\to\infty)$をとると（左辺は区分求積に注意），

$$
\int_0^1 f(x)\,dx\leqq f\left(\frac{1}{2}\right)
$$

（証明終）