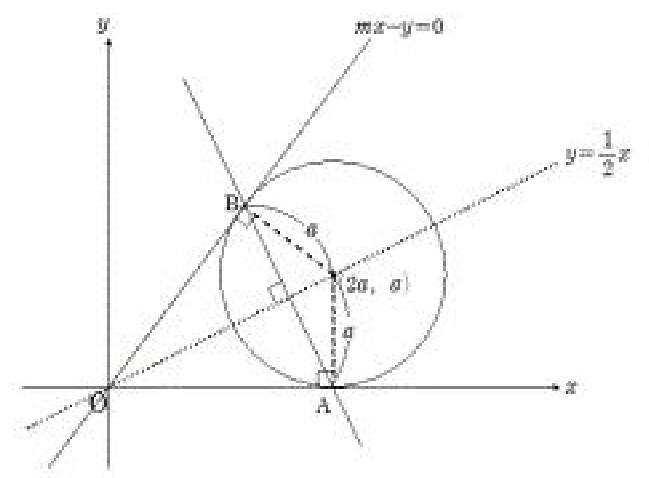
$\boxed{\text{I}}$

(1) $x^2+y^2-4ax-2ay+4a^2=0$

$\iff (x-2a)^2+(y-a)^2=a^2$　　円の中心座標　$\underline{(2a,\ a)}$　半径　$a\ (>0)$

ここで，

$$\begin{cases} x=2a \\ y=a \end{cases} \qquad \therefore\ y=\dfrac{\overset{(ア)}{\boxed{1}}}{\underset{(イ)}{\boxed{2}}}x \qquad (\text{答})$$



$$\frac{|2am-a|}{\sqrt{m^2+1}}=a$$

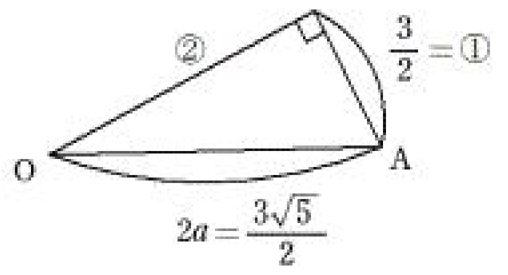
$\iff |2m-1|=\sqrt{m^2+1}$

$\therefore\ 4m^2-4m+1=m^2+1$

$\iff 3m\left(m-\dfrac{4}{3}\right)=0 \quad \therefore\ m=\overset{(ウ)}{\boxed{0}},\ \dfrac{\overset{(エ)}{\boxed{4}}}{\underset{(オ)}{\boxed{3}}}$　(答)

$(\text{ABの傾き})=\overset{(カキ)}{\boxed{-2}}$　(答)

右図より，



$2a=\dfrac{3\sqrt{5}}{2} \qquad \therefore\ a=\dfrac{\overset{(ク)}{\boxed{3}}\sqrt{\overset{(ケ)}{\boxed{5}}}}{\underset{(コ)}{\boxed{4}}}$　(答)

(2)

(a)

$13=11\cdot 1+2$

$13^2=(11\cdot 1+2)^2=11\cdot M+4$ (M：整数)

$1020=11\cdot 92+8$

$1020^3=(11\cdot 92+8)^3=11\cdot N+512$ (N：整数)

$\quad =11\cdot(N+46)+\overset{(ア)}{\boxed{6}}$　(答)

(b)

$\begin{cases} a=Ga' \\ b=Gb' \end{cases}$　(a'とb'は互いに素な整数)

$G=1020^3,\ Ga'b'=1020^{10}$

ここで，

$ab=G\cdot Ga'b'$

$\quad =1020^3\cdot 1020^{10}$

$\quad =1020^{13}$

$\quad =(1020^3)^4\cdot 1020$

$\quad =(11\cdot N'+6)^4\cdot(11\cdot 92+8)$ (N': 整数)

$\quad =11\cdot N''+6^4\cdot 8$ (N'': 整数)

$\quad =11\cdot(N''+942)+\overset{(イ)}{\boxed{6}}$　(答)

$(a+b)^2=a^2+b^2+2ab$

$\quad =11\cdot M'+10+2(11\cdot N'''+6)$　($M',\ N'''$: 整数)

$\quad =11\cdot(M'+2N'''+2)+\overset{(ウ)}{\boxed{0}}$　(答)

$(a-b)^2=a^2+b^2-2ab$

$\quad =11\cdot(M'-N''')-2$

$\quad =11\cdot(M'-2N'''-2)+\overset{(エ)}{\boxed{9}}$　(答)

$m=0,1,2,3,4,5,6,7,8,9,10$とするとき，　m^2を11で割った余りは

0, 1, 4, 9, 5, 3, 3, 5, 9, 4, 1なので，

$\begin{cases} a+b=11\cdot M''+0 \\ a-b=11\cdot M'''+3 \ \text{ or } \ 11\cdot M''''+8 \end{cases}$

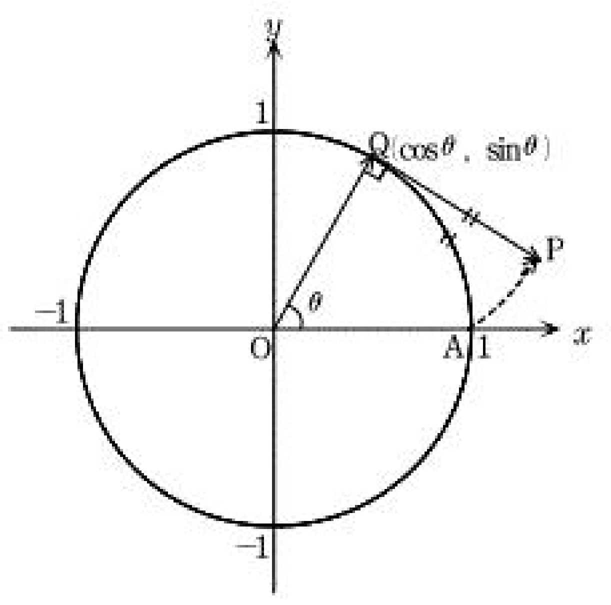
$\therefore$　a，bを11で割った余りは$\overset{(オ)}{\boxed{4}},\ \overset{(カ)}{\boxed{7}}$　(答)

(3) $\overset{\frown}{\mathrm{AQ}}=|\overrightarrow{\mathrm{PQ}}|=\theta$より，

$\overrightarrow{\mathrm{OP}}=(x(\theta),\ y(\theta))=\overrightarrow{\mathrm{OQ}}+\overrightarrow{\mathrm{QP}}$

$\quad =(\cos\theta,\ \sin\theta)+\theta(\sin\theta,\ -\cos\theta)$

$\quad =(\cos\theta,\ \sin\theta)+\theta\left(\cos\left(\theta-\dfrac{\pi}{2}\right),\ \sin\left(\theta-\dfrac{\pi}{2}\right)\right)$



$\therefore\ a=\dfrac{\overset{(アイ)}{\boxed{-1}}}{\underset{(ウ)}{\boxed{2}}}$　(答)

$$\begin{cases} x'(\theta)=-\sin\theta+\cos\left(\theta-\dfrac{\pi}{2}\right)-\theta\sin\left(\theta-\dfrac{\pi}{2}\right) \\ \quad =-\sin\theta+\sin\theta+\theta\cos\theta \\ \quad =\theta\cos\theta \\ y'(\theta)=\cos\theta+\sin\left(\theta-\dfrac{\pi}{2}\right)-\theta\cos\left(\theta-\dfrac{\pi}{2}\right) \\ \quad =\cos\theta-\cos\theta+\theta\sin\theta \\ \quad =\theta\sin\theta \end{cases}$$

ここで，

$|\vec{v}|^2=\theta^2(\cos^2\theta+\sin^2\theta)=\theta^2$

$\therefore\ \int_0^{\alpha}\sqrt{\theta^2}\,d\theta=\int_0^{\alpha}\theta\,d\theta=\dfrac{1}{2}\alpha^2$　より，　$b=\dfrac{\overset{(エ)}{\boxed{1}}}{\underset{(オ)}{\boxed{2}}},\ c=\overset{(カ)}{\boxed{2}}$　(答)

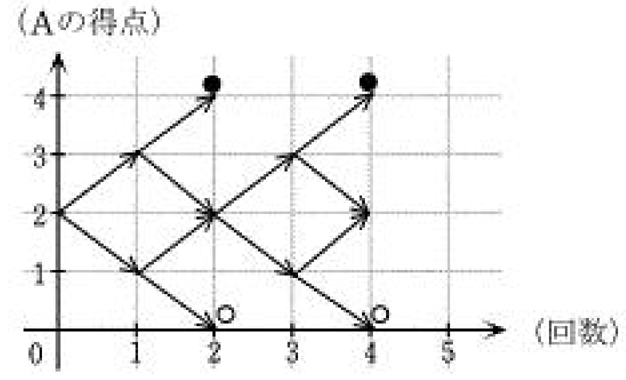
$\alpha=2\pi$代入，　$\dfrac{1}{2}(2\pi)^2=2\pi^2$より，

$d=\overset{(キ)}{\boxed{2}},\ f=\overset{(ク)}{\boxed{2}}$　(答)

Ⅱ

(1)

$$p(2\text{回},Ⓐ)=\left(\frac{1}{3}\right)^2=\frac{\overset{(ア)}{\boxed{1}}}{\underset{(イ)}{\boxed{9}}}\qquad(答)$$



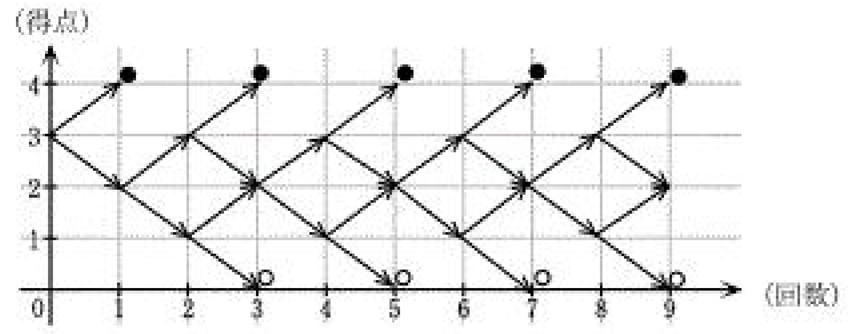
$$p(4\text{回},Ⓐ)=\left(\frac{1}{3}\cdot\frac{2}{3}+\frac{2}{3}\cdot\frac{1}{3}\right)\cdot\left(\frac{1}{3}\right)^2=\frac{\overset{(ウ)}{\boxed{4}}}{\underset{(エオ)}{\boxed{81}}}\qquad(答)$$

$$q_{2k}=\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\cdot\frac{2}{3}+\frac{2}{3}\cdot\frac{1}{3}\right)^{k-1}$$

$$=\frac{1}{9}\left(\frac{4}{9}\right)^{k-1}$$

$$\therefore\ \sum_{k=1}^{\infty}q_{2k}=\sum_{k=1}^{\infty}\frac{1}{9}\left(\frac{4}{9}\right)^{k-1}=\frac{\frac{1}{9}}{1-\frac{4}{9}}=\frac{\overset{(カ)}{\boxed{1}}}{\underset{(キ)}{\boxed{5}}}\qquad(答)$$

(2)



$$p(1\text{回},Ⓐ)+p(3\text{回},Ⓐ)+p(5\text{回},Ⓐ)+p(7\text{回},Ⓐ)+\cdots\cdots$$

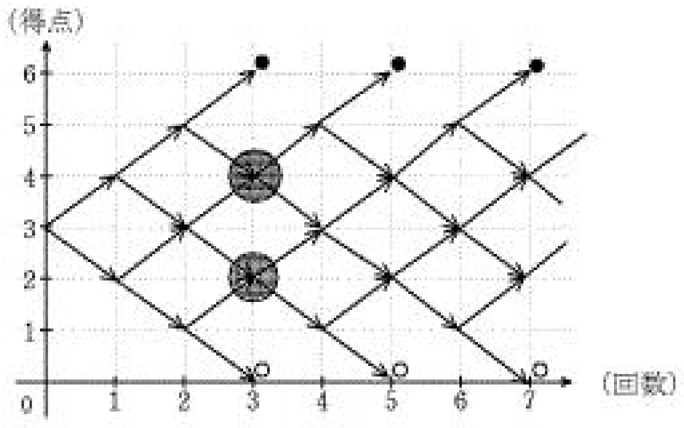
$$=\frac{1}{3}+\frac{2}{3}\cdot\frac{1}{5}\ \leftarrow\ \left((1)\text{の}\ \sum_{k=1}^{\infty}q_{2k}=\frac{1}{5}\ \text{を利用}\right)$$

$$=\frac{\overset{(ク)}{\boxed{7}}}{\underset{(ケコ)}{\boxed{15}}}\qquad(答)$$

$$\frac{1}{3}\cdot\sum_{k=1}^{\infty}q_{2k}=\frac{1}{3}\cdot\frac{1}{5}=\frac{\overset{(サ)}{\boxed{1}}}{\underset{(シス)}{\boxed{15}}}\qquad(答)$$

(3)

$${}_3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1=\frac{6}{27}=\frac{\overset{(セ)}{\boxed{2}}}{\underset{(ソ)}{\boxed{9}}}\qquad(答)$$



$${}_3C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2=\frac{12}{27}=\frac{\overset{(タ)}{\boxed{4}}}{\underset{(チ)}{\boxed{9}}}\qquad(答)$$

Aが n 点からの優勝する確率を q_n とすると，
n=1，2，3，4，5 に対して，

$$q_n=\frac{1}{3}q_{n+1}+\frac{2}{3}q_{n-1}$$

$$\iff\ q_{n+1}-3q_n+2q_{n-1}=0$$

$$\iff\ \begin{cases}q_{n+1}-q_n=2(q_n-q_{n-1})\\ q_{n+1}-2q_n=1(q_n-2q_{n-1})\end{cases}$$

$$\therefore\ \begin{cases}q_{n+1}-q_n=(q_1-q_0)\cdot 2^n\\ q_{n+1}-2q_n=(q_1-2q_0)\cdot 1^n\end{cases}\qquad\therefore\ q_n=\alpha\cdot 2^n+\beta\ \left(\text{適当な実数}\ \alpha,\beta\ \text{を用いて}\right)$$

ここで，

$$\begin{cases}q_0=\alpha+\beta=0\\ q_6=64\alpha+\beta=1\end{cases}\quad\therefore\ (\alpha,\beta)=\left(\frac{1}{63},-\frac{1}{63}\right)$$

これより，

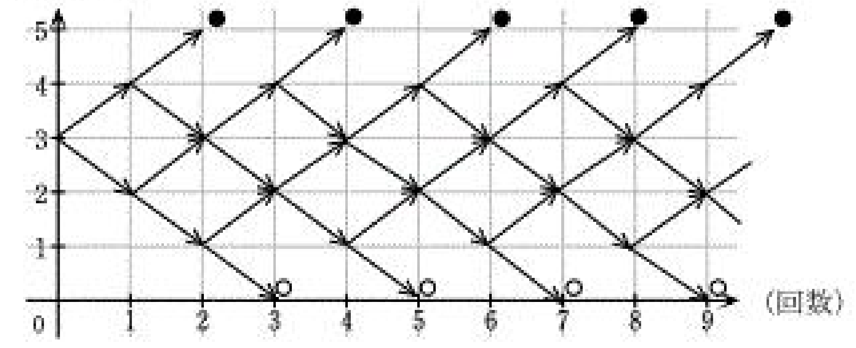
$$q_n=\frac{1}{63}(2^n-1)\ \ (n=1,2,3,4,5)$$

と表せるので，

$$q_3=\frac{7}{63}=\frac{\overset{(ツ)}{\boxed{1}}}{\underset{(テト)}{\boxed{9}}}\qquad(答)$$

(4)

$$\underset{Ⓐ}{\frac{1}{2}}\cdot\left(\underset{Ⓐ}{\frac{1}{2}}+\underset{Ⓑ}{\frac{1}{2}}p\right)$$



$$=\frac{\overset{(ト)}{\boxed{1}}}{\underset{(ナ)}{\boxed{4}}}p+\frac{\overset{(ニ)}{\boxed{1}}}{\underset{(ヌ)}{\boxed{4}}}\qquad(答)$$

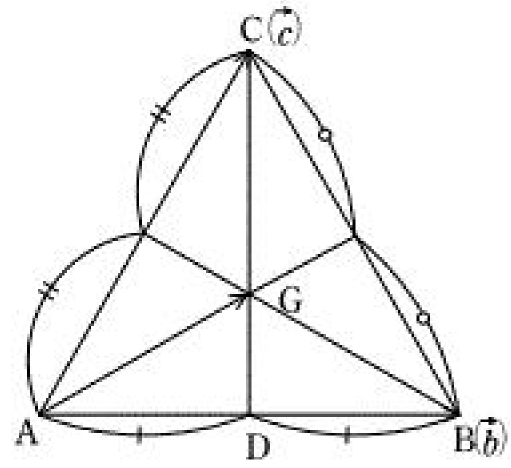
$$p=\left(\frac{1}{4}p+\frac{1}{4}\right)+\underset{Ⓑ}{\frac{1}{2}}\underline{(1-p)}$$

(上記のグラフより)

$$\iff\ p=\frac{1}{4}p+\frac{1}{4}+\frac{1}{2}-\frac{1}{2}p\qquad\therefore\ p=\frac{\overset{(ネ)}{\boxed{3}}}{\underset{(ノ)}{\boxed{5}}}\qquad(答)$$

$\boxed{\text{III}}$

(1) $\overrightarrow{AG}=k\cdot\dfrac{\overrightarrow{AB}+\overrightarrow{AC}}{2}=\dfrac{k}{2}\vec{b}+\dfrac{k}{2}\vec{c}$ (k：実数)……①



さらに，

$$\overrightarrow{AG}=(1-t)\overrightarrow{AC}+t\cdot\frac{1}{2}\overrightarrow{AB}$$
$$=\frac{t}{2}\vec{b}+(1-t)\vec{c} \qquad (t:\text{実数})\cdots\cdots②$$

$\vec{b}\neq\vec{0}, \vec{c}\neq\vec{0}, \vec{b}\nparallel\vec{c}$ なので，①，②より，

$$\begin{cases}\dfrac{k}{2}=\dfrac{t}{2}\\[2mm] \dfrac{k}{2}=1-t\end{cases} \qquad \therefore\ k=t=\frac{2}{3}$$

$$\therefore\ \overrightarrow{AG}=\frac{1}{3}\left(\vec{b}+\vec{c}\right) \qquad (\text{答})$$

(2) $\overrightarrow{AB}\cdot\overrightarrow{AC}=\underline{\vec{b}\cdot\vec{c}=\left|\vec{b}\right|\left|\vec{c}\right|\cos\theta}$ ……③

ここで， $\overrightarrow{DA}=-\dfrac{1}{2}\vec{b}$, $\overrightarrow{DG}=\overrightarrow{AG}-\overrightarrow{AD}=\dfrac{1}{3}\left(\vec{b}+\vec{c}\right)-\dfrac{1}{2}\vec{b}=\dfrac{1}{3}\vec{c}-\dfrac{1}{6}\vec{b}$

$$\therefore\ \overrightarrow{DA}\cdot\overrightarrow{DG}=-\frac{1}{2}\vec{b}\cdot\left(\frac{1}{3}\vec{c}-\frac{1}{6}\vec{b}\right)$$
$$=-\frac{1}{6}\vec{b}\cdot\vec{c}+\frac{1}{12}\left|\vec{b}\right|^2$$

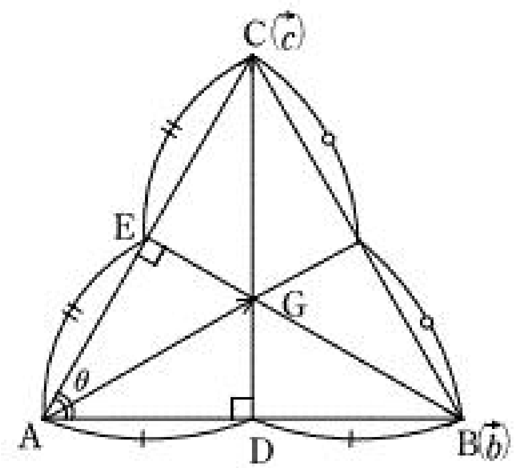
(③より)

$$=\frac{1}{12}\left|\vec{b}\right|^2-\frac{1}{6}\left|\vec{b}\right|\left|\vec{c}\right|\cos\theta \qquad (\text{答})$$

(3)

「重心」≡「外心」のとき，

(線分 AC の中点を E とする)



$$\begin{cases}\overrightarrow{DA}\cdot\overrightarrow{DG}=0\\ \overrightarrow{EA}\cdot\overrightarrow{EG}=0\end{cases}$$

$$\Longleftrightarrow \begin{cases}\left|\vec{b}\right|^2=2\left|\vec{b}\right|\left|\vec{c}\right|\cos\theta\\ \left|\vec{c}\right|^2=2\left|\vec{c}\right|\left|\vec{b}\right|\cos\theta\end{cases}$$

$$\Longleftrightarrow \begin{cases}\left|\vec{b}\right|=\left|\vec{c}\right|\\ \theta=\dfrac{\pi}{3}\end{cases}$$

三角形 ABC において，重心と外心が一致する必要十分条件は，

<u>三角形 ABC が正三角形</u>であること． (答)