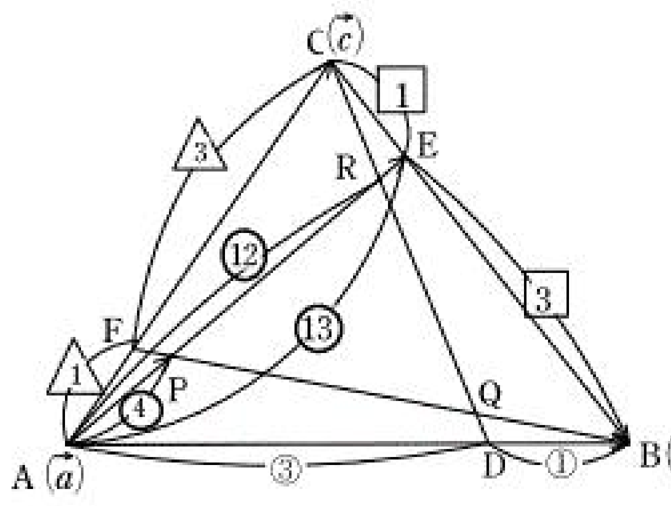
$\boxed{\text{I}}$

(1)

$$\overrightarrow{AE}=\frac{\vec{b}+3\vec{c}}{3+1}=\frac{\overset{(ア)}{\boxed{1}}}{\underset{(イ)}{\boxed{4}}}\vec{b}+\frac{\overset{(ウ)}{\boxed{3}}}{\underset{(エ)}{\boxed{4}}}\vec{c}\qquad\cdots(答)$$

$$\overrightarrow{AP}=k\overrightarrow{AE}=\frac{k}{4}\overrightarrow{AB}+\frac{3k}{4}\overrightarrow{AC}\ \ (k\text{は実数})$$
$$=\frac{k}{4}\overrightarrow{AB}+3k\overrightarrow{AF}$$



ここで,

$$\frac{k}{4}+3k=1\qquad\therefore\ k=\frac{4}{13}$$

$$\therefore\ \overrightarrow{AP}=\frac{\overset{(オ)}{\boxed{1}}}{\underset{(カキ)}{\boxed{13}}}\vec{b}+\frac{\overset{(ク)}{\boxed{3}}}{\underset{(ケコ)}{\boxed{13}}}\vec{c}\qquad\cdots(答)$$

同様に,

$$\overrightarrow{AR}=l\overrightarrow{AE}=\frac{l}{4}\overrightarrow{AB}+\frac{3l}{4}\overrightarrow{AC}\ \ (l\text{は実数})$$
$$=\frac{l}{3}\overrightarrow{AD}+\frac{3l}{4}\overrightarrow{AC}$$

ここで,

$$\frac{l}{3}+\frac{3l}{4}=1\qquad\therefore\ l=\frac{12}{13}$$

$$\therefore\ AP:PR:RE=\overset{(サ)}{\boxed{4}}:\overset{(シ)}{\boxed{8}}:\overset{(ス)}{\boxed{1}}\qquad\cdots(答)$$

$$\triangle PQR=\left(1-3\cdot\frac{3}{4}\cdot\frac{4}{13}\right)\triangle ABC$$
$$=\frac{\overset{(セ)}{\boxed{4}}}{\underset{(ソタ)}{\boxed{13}}}\triangle ABC\qquad\cdots(答)$$

$\triangle ABC$ の重心と, $\triangle PQR$ の重心は一致.

$$\therefore\ \overrightarrow{AG}=\frac{\overset{(チ)}{\boxed{1}}}{\underset{(ツ)}{\boxed{3}}}\vec{b}+\frac{\overset{(テ)}{\boxed{1}}}{\underset{(ト)}{\boxed{3}}}\vec{c}\qquad\cdots(答)$$

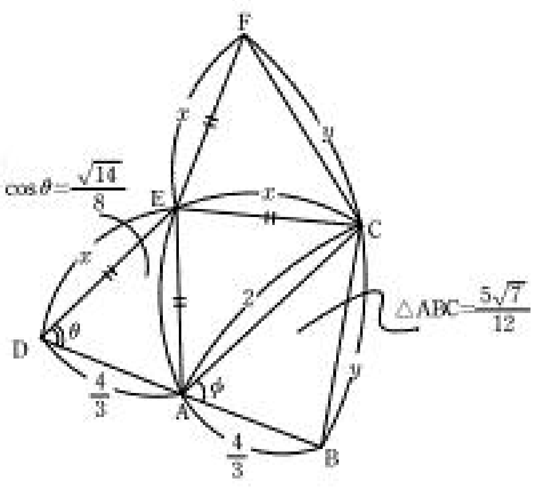
(2)

(i) $DE=EA=EC=EF=x$ とおく.

三角形 ADE において,

$$x^2+\left(\frac{4}{3}\right)^2-2\cdot x\cdot\frac{4}{3}\cdot\frac{\sqrt{14}}{8}=x^2$$

$$\therefore\ x=\frac{8\sqrt{14}}{21}\qquad\therefore\ DE=\frac{\overset{(ア)}{\boxed{8}}\sqrt{\overset{(イウ)}{\boxed{14}}}}{\underset{(エオ)}{\boxed{21}}}\qquad\cdots(答)$$



$\angle BAC=\varphi$ とする.

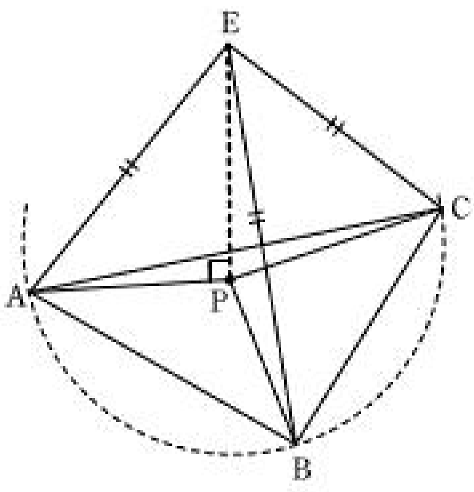
$$\triangle ABC=\frac{1}{2}\cdot\frac{4}{3}\cdot 2\sin\varphi=\frac{5\sqrt{7}}{12}\qquad\therefore\ \sin\varphi=\frac{\overset{(カ)}{\boxed{5}}\sqrt{\overset{(キ)}{\boxed{7}}}}{\underset{(クケ)}{\boxed{16}}}\qquad\cdots(答)$$

$$\cos\varphi=\sqrt{1-\sin^2\varphi}=\sqrt{\frac{81}{256}}=\frac{9}{16}$$

三角形 ABC において,

$$y^2=2^2+\left(\frac{4}{3}\right)^2-2\cdot 2\cdot\frac{4}{3}\cdot\frac{9}{16}=\frac{25}{9}$$

$$\therefore\ y=\frac{5}{3}\quad\therefore\ BC=\frac{\overset{(コ)}{\boxed{5}}}{\underset{(サ)}{\boxed{3}}}\qquad\cdots(答)$$

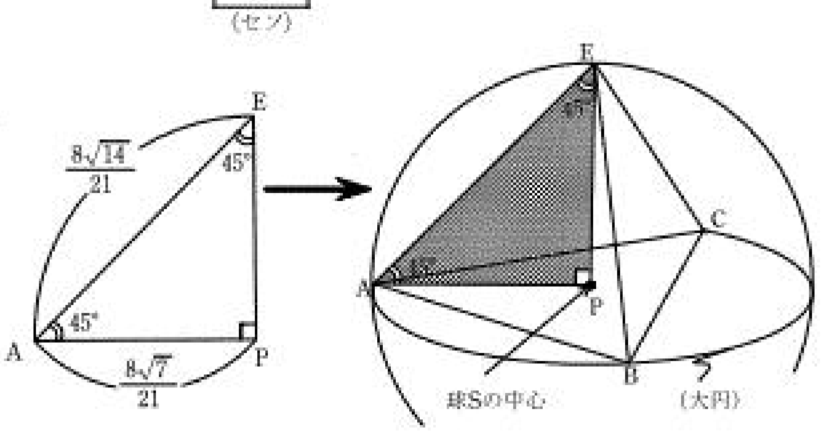


$$\frac{\frac{5}{3}}{\sin\varphi}=2R\qquad\therefore\ R=\frac{8\sqrt{7}}{21}\qquad\therefore\ AP=\frac{\overset{(シ)}{\boxed{8}}\sqrt{\overset{(ス)}{\boxed{7}}}}{\underset{(セソ)}{\boxed{21}}}\qquad\cdots(答)$$

(ii) $AE=\frac{8\sqrt{14}}{21}$ と $AP=\frac{8\sqrt{7}}{21}$ より, 右図の通り.

これより,

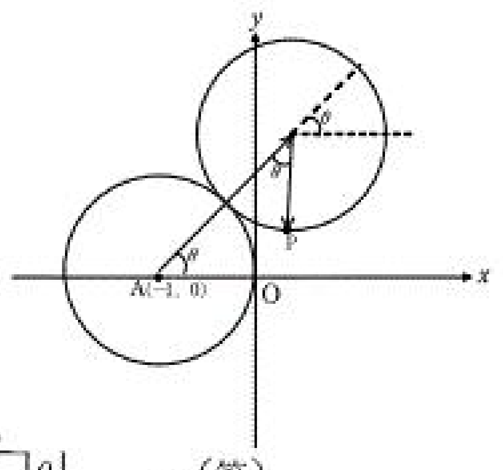
$$(\text{半径})=\frac{\overset{(タ)}{\boxed{8}}\sqrt{\overset{(チ)}{\boxed{7}}}}{\underset{(ツテ)}{\boxed{21}}}\qquad\cdots(答)$$



(3)

$$\overrightarrow{OQ}=(-1,\,0)+2(\cos\theta,\,\sin\theta)$$

$$\therefore\ Q\left(\overset{(ア)}{\boxed{2}}\cos\theta-\overset{(イ)}{\boxed{1}},\ \overset{(ア)}{\boxed{2}}\sin\theta\right)\qquad\cdots(答)$$



$$\overrightarrow{OP}=\overrightarrow{OQ}+(\cos(2\theta-\pi),\,\sin(2\theta-\pi))$$
$$=(2\cos\theta-1,\,2\sin\theta)+(-\cos 2\theta,\,-\sin 2\theta)$$
$$=\left(\overset{(ア)}{\boxed{2}}\cos\theta-\cos\overset{(ウ)}{\boxed{2}}\theta-\overset{(イ)}{\boxed{1}},\ \overset{(ア)}{\boxed{2}}\sin\theta-\sin\overset{(ウ)}{\boxed{2}}\theta\right)\qquad\cdots(答)$$
$$=2(1-\cos\theta)(\cos\theta,\,\sin\theta)$$

$$\therefore\ r(\theta)=2(1-\cos\theta)=\overset{(エ)}{\boxed{2}}-\overset{(オ)}{\boxed{2}}\cos\theta\qquad\cdots(答)$$

$$S=\frac{1}{2}\int_0^{2\pi}4(1-\cos\theta)^2d\theta$$
$$=\int_0^{2\pi}(3-4\cos\theta+\cos 2\theta)d\theta$$
$$=\left[3\theta-4\sin\theta+\frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$
$$=\overset{(カ)}{\boxed{6}}\pi\qquad\cdots(答)$$

$$L=\int_0^{2\pi}\sqrt{4(1-2\cos\theta+\cos^2\theta)+4\sin^2\theta}\,d\theta$$
$$=\int_0^{2\pi}\sqrt{8-8\cos\theta}\,d\theta$$
$$=\int_0^{2\pi}\sqrt{16\sin^2\frac{\theta}{2}}\,d\theta$$
$$=4\int_0^{2\pi}\left|\sin\frac{\theta}{2}\right|d\theta$$
$$=8\left[-\cos\frac{\theta}{2}\right]_0^{2\pi}$$
$$=\overset{(キク)}{\boxed{16}}\qquad\cdots(答)$$

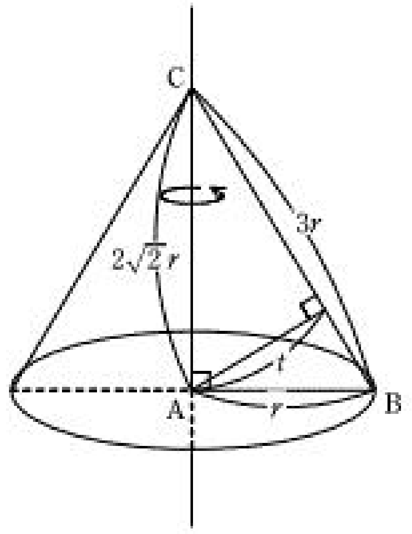
$\boxed{\text{II}}$

(1)

$$t = r \cdot \frac{2\sqrt{2}r}{3r} = \frac{\overset{(ア)}{\boxed{2}}\sqrt{\overset{(イ)}{\boxed{2}}}}{\underset{(ウ)}{\boxed{3}}}r \quad \cdots(答)$$

$$V = \frac{1}{3}\cdot\pi r^2\cdot 2\sqrt{2}r = \frac{2\sqrt{2}}{3}\pi\left(\frac{3t}{2\sqrt{2}}\right)^3 = \frac{\overset{(エ)}{\boxed{9}}}{\underset{(オ)}{\boxed{8}}}\pi t^3 \quad \cdots(答)$$

$$S = \frac{dV}{dt} + \pi r^2 = \frac{27}{8}\pi t^2 + \frac{9}{8}\pi t^2 = \frac{\overset{(カ)}{\boxed{9}}}{\underset{(キ)}{\boxed{2}}}\pi t^2 \quad \cdots(答)$$



(2)

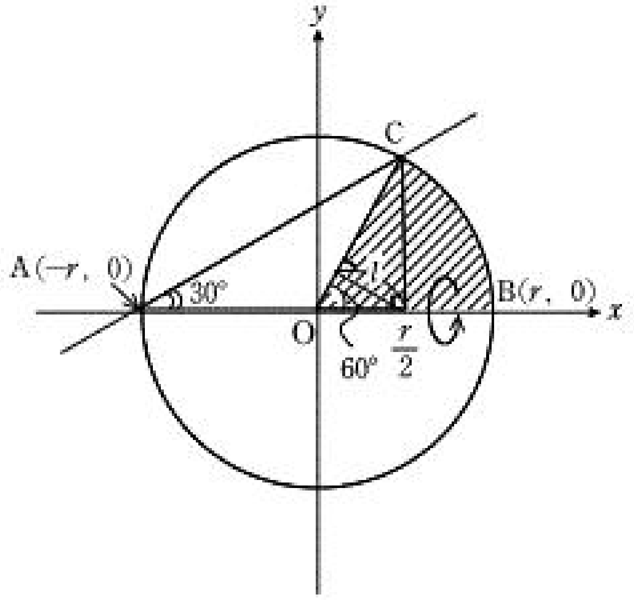
$$C\left(\frac{\overset{(ク)}{\boxed{1}}}{\underset{(ケ)}{\boxed{2}}}r,\ \frac{\sqrt{\overset{(コ)}{\boxed{3}}}}{\underset{(サ)}{\boxed{2}}}r\right) \quad \cdots(答)$$

$$V' = \frac{1}{3}\cdot\pi\left(\frac{\sqrt{3}}{2}r\right)^2\cdot\frac{r}{2} + \pi\int_{\frac{r}{2}}^{r}(r^2 - x^2)\,dx$$

$$= \frac{1}{8}\pi r^3 + \pi\left[r^2x - \frac{x^3}{3}\right]_{\frac{r}{2}}^{r}$$

$$= \frac{1}{8}\pi r^3 + \frac{5}{24}\pi r^3$$

$$= \frac{\overset{(シ)}{\boxed{1}}}{\underset{(ス)}{\boxed{3}}}\pi r^3 \quad \cdots(答)$$



右図の　$l = \frac{r}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}r \quad \left(\frac{dr}{dl} = \frac{4}{\sqrt{3}}\right)$

$$S' = \frac{dV'}{dr} + \left(\frac{1}{8}\pi r^3\right)'\cdot\frac{dr}{dl}$$

$$= \pi r^2 + \frac{3}{8}\pi r^2\cdot\frac{4}{\sqrt{3}}$$

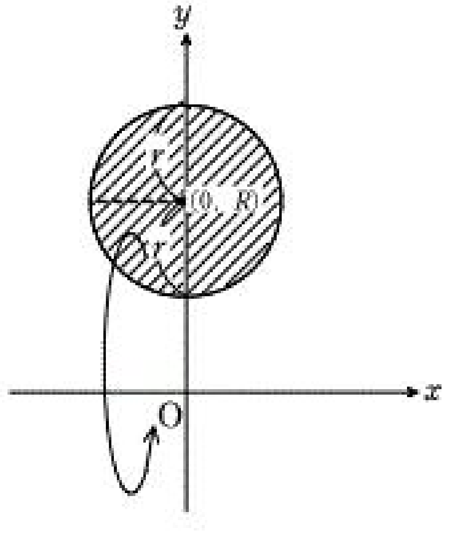
$$= \frac{\overset{(セ)}{\boxed{2}} + \sqrt{\overset{(ソ)}{\boxed{3}}}}{\underset{タ}{\boxed{2}}}\pi r^2 \quad \cdots(答)$$

(3)　$V'' = \pi r^2\cdot 2\pi R = 2\pi^2r^2R$ より，

$$a = \overset{(チ)}{\boxed{2}},\ b = \overset{(ツ)}{\boxed{2}},\ c = \overset{(テ)}{\boxed{2}},\ d = \overset{(ト)}{\boxed{1}} \quad \cdots(答)$$

$S'' = \frac{dV''}{dr} = 4\pi^2 rR$ より，

$$s = \overset{(ナ)}{\boxed{4}},\ t = \overset{(ニ)}{\boxed{2}},\ u = \overset{(ヌ)}{\boxed{1}},\ v = \overset{(ネ)}{\boxed{1}} \quad \cdots(答)$$



Ⅲ

(1)　$x>0$ において，十分大きな n に対し，

$$e^x > 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots+\frac{x^n}{n!} > \frac{x^3}{3!}$$

これより，　$0<\dfrac{x^2}{e^x}<\dfrac{x^2}{\dfrac{x^3}{3!}}$

$$(\text{最右辺})=0 \quad (x\to\infty)$$

はさみうちの原理より，

$$\lim_{x\to\infty}\frac{x^2}{e^x}=0$$

…(答)

(2)　$y=(1+x)e^x$ 上の点 $\mathrm{P}\left(t,(1+t)e^t\right)$ 上での接線は，$y'=e^x+(1+x)e^x=(2+x)e^x$ より，

$$y=(2+t)e^t(x-t)+(1+t)e^t$$

$$\Longleftrightarrow \quad y=(2+t)e^t x+\left(1-t-t^2\right)e^t$$

これが $(0,a)$ を通るから，　$a=\left(1-t-t^2\right)e^t$　……①

「求める接線の本数」　$\Longleftrightarrow$　「①を満たす異なる実数解の個数」

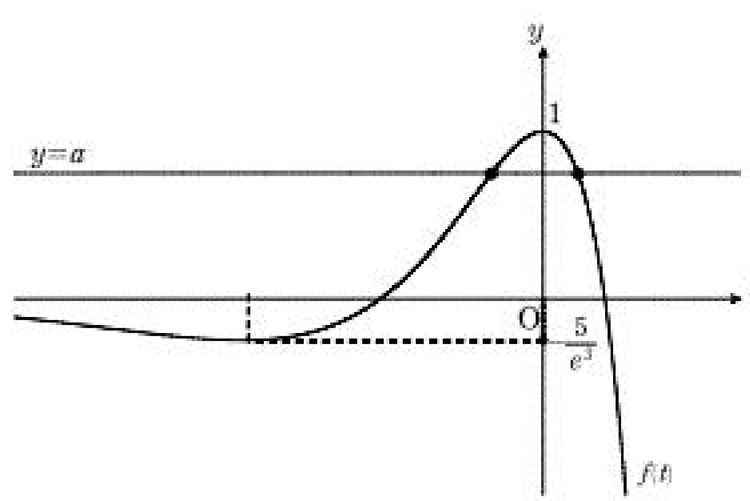
ここで，2 関数 $\begin{cases} f(t)=\left(1-t-t^2\right)e^t \\ y=a \end{cases}$ に分離して，その共有点の個数より調べる．

$$\begin{aligned} f'(t)&=(-1-2t)e^t+\left(1-t-t^2\right)e^t \\ &=\left(-3t-t^2\right)e^t \\ &=-t(t+3)e^t \end{aligned}$$

t	$(-\infty)$	$\cdots$	-3	$\cdots$	0	$\cdots$	$(+\infty)$
$f'(t)$		$-$	0	$+$	0	$-$	
$f(t)$	(0)	↘	$-\dfrac{5}{e^3}$	↗	1	↘	$(-\infty)$

↑

(1)より



右図より，求める接線の本数は，

$$\begin{cases} a>1 & \cdots\cdots 0\,(\text{本}) \\ a=1,\ a<-\dfrac{5}{e^3} & \cdots\cdots 1\,(\text{本}) \\ 1>a\geqq 0,\ a=-\dfrac{5}{e^3} & \cdots\cdots 2\,(\text{本}) \\ 0>a>-\dfrac{5}{e^3} & \cdots\cdots 3\,(\text{本}) \end{cases}$$

…(答)