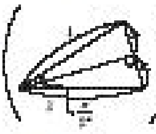
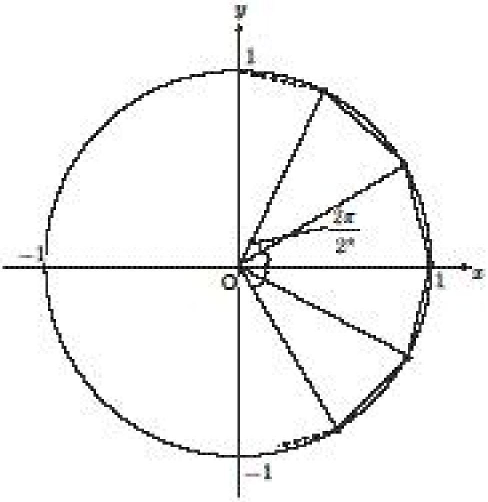
$\boxed{1}$

(1)

$$S_n = 2^n \cdot \left(\frac{1}{2}\cdot 1^2 \cdot \sin\frac{2\pi}{2^n}\right)$$
$$= \underline{2^{n-1}\sin\frac{\pi}{2^{n-1}}}$$

$L_n = 2^n \cdot \Big($  $\Big)$

$$= 2^n \cdot \left(2\sin\frac{\pi}{2^n}\right)$$
$$= \underline{2^{n+1}\sin\frac{\pi}{2^n}}$$



(2)　(1)を用いて

$$\frac{S_n}{S_{n+1}} = \frac{2^{n-1}\sin\frac{\pi}{2^{n-1}}}{2^n\sin\frac{\pi}{2^n}} = \frac{2^{n-1}\sin 2\cdot\frac{\pi}{2^n}}{2^n\sin\frac{\pi}{2^n}}$$
$$= \frac{2^n\sin\frac{\pi}{2^n}\cos\frac{\pi}{2^n}}{2^n\sin\frac{\pi}{2^n}} = \underline{\cos\frac{\pi}{2^n}}$$

$$\frac{S_n}{L_n} = \frac{2^{n-1}\sin\frac{\pi}{2^{n-1}}}{2^{n+1}\sin\frac{\pi}{2^n}} = \frac{2^{n-1}\sin 2\cdot\frac{\pi}{2^n}}{2^{n+1}\sin\frac{\pi}{2^n}}$$
$$= \frac{2^n\sin\frac{\pi}{2^n}\cos\frac{\pi}{2^n}}{2^{n+1}\sin\frac{\pi}{2^n}} = \underline{\frac{1}{2}\cos\frac{\pi}{2^n}}$$

(3)

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2^{n-1}\sin\frac{\pi}{2^{n-1}} = \lim_{n\to\infty} \underbrace{\frac{\sin\frac{\pi}{2^{n-1}}}{\frac{\pi}{2^{n-1}}}}_{\to 1}\cdot\frac{2^{n-1}\cdot\pi}{2^{n-1}} = \underline{\pi}$$

$$\lim_{n\to\infty}\cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cos\frac{\pi}{2^4}\cdots\cdots\cos\frac{\pi}{2^n}$$
$$= \lim_{n\to\infty}\frac{S_2}{S_3}\cdot\frac{S_3}{S_4}\cdot\frac{S_4}{S_5}\cdot\cdots\cdots\cdot\frac{S_n}{S_{n+1}} \qquad \left(\because\ \frac{S_n}{S_{n+1}} = \cos\frac{\pi}{2^n}\right)$$
$$= \lim_{n\to\infty}\frac{S_2}{S_{n+1}}$$
$$= \lim_{n\to\infty}\frac{2}{S_{n+1}} = \underline{\frac{2}{\pi}} \qquad \left(\because\ \lim_{n\to\infty}S_n = \pi\right)$$

(4)

$$2^n\frac{S_2}{L_2}\cdot\frac{S_3}{L_3}\cdot\frac{S_4}{L_4}\cdot\cdots\cdot\frac{S_n}{L_n}$$
$$= 2^n\underbrace{\left(\frac{1}{2}\cos\frac{\pi}{2^2}\right)\cdot\left(\frac{1}{2}\cos\frac{\pi}{2^3}\right)\cdot\left(\frac{1}{2}\cos\frac{\pi}{2^4}\right)\cdot\cdots\cdot\left(\frac{1}{2}\cos\frac{\pi}{2^n}\right)}_{(n-1)\text{個}}$$
$$= 2\cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cdot\cos\frac{\pi}{2^4}\cdot\cdots\cdot\cos\frac{\pi}{2^n}$$

よって、

$$\lim_{n\to\infty}2^n\frac{S_2}{L_2}\cdot\frac{S_3}{L_3}\cdot\frac{S_4}{L_4}\cdot\cdots\cdot\frac{S_n}{L_n}$$
$$= \lim_{n\to\infty}2\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot\cos\frac{\pi}{2^4}\cdot\cdots\cdot\cos\frac{\pi}{2^n}$$
$$= 2\cdot\frac{2}{\pi} = \underline{\frac{4}{\pi}} \quad \left(\because\ \lim_{n\to\infty}\cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cos\frac{\pi}{2^4}\cdot\cdots\cdot\cos\frac{\pi}{2^n} = \frac{2}{\pi}\right)$$

$\boxed{2}$

(1) $P_0(5+s_0,\ -s_0,\ 0)$, $Q_0(t_0,\ 0,\ 2+2t_0)$ （ただし，s_0, t_0 は実数）とおける．

ここで，$\overrightarrow{P_0Q_0}=(t_0-s_0-5,\ s_0,\ 2t_0+2)$ となるので，

$$\begin{cases}\overrightarrow{P_0Q_0}\cdot(1,-1,0)=t_0-s_0-5-s_0=0\\ \overrightarrow{P_0Q_0}\cdot(1,0,2)=t_0-s_0-5+4t_0+4=0\end{cases}\iff\begin{cases}t_0-2s_0=5\\ 5t_0-s_0=1\end{cases}$$

$$\therefore\ s_0=-\frac{8}{3},\ t_0=-\frac{1}{3}$$

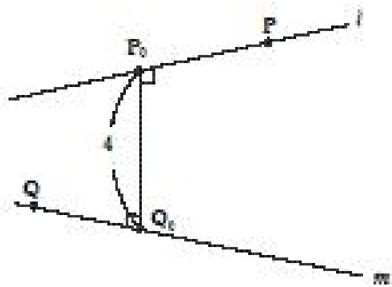
$$\underline{P_0\left(\frac{7}{3},\frac{8}{3},0\right),\ Q_0\left(-\frac{1}{3},0,\frac{4}{3}\right)}$$

(2)

$$\overrightarrow{P_0Q_0}=\left(-\frac{8}{3},-\frac{8}{3},\frac{4}{3}\right)=\frac{4}{3}(-2,-2,1)\ \text{より，}$$

$$\left|\overrightarrow{P_0Q_0}\right|=\frac{4}{3}\cdot\sqrt{4+4+1}=4\qquad\qquad \underline{\left|\overrightarrow{P_0Q_0}\right|=4}$$

(3) $\underline{\overrightarrow{PQ}=\overrightarrow{PP_0}+\overrightarrow{P_0Q_0}+\overrightarrow{Q_0Q}}$



これより，

$$\begin{aligned}\left|\overrightarrow{PQ}\right|^2&=\left|\left(\overrightarrow{PP_0}+\overrightarrow{Q_0Q}\right)+\overrightarrow{P_0Q_0}\right|^2\\&=\left|\overrightarrow{PP_0}+\overrightarrow{Q_0Q}\right|^2+2\left(\overrightarrow{PP_0}+\overrightarrow{Q_0Q}\right)\cdot\overrightarrow{P_0Q_0}+16\\&=\left|\overrightarrow{PP_0}+\overrightarrow{Q_0Q}\right|^2+2\left(\underbrace{\overrightarrow{PP_0}\cdot\overrightarrow{P_0Q_0}}_{0}+\underbrace{\overrightarrow{Q_0Q}\cdot\overrightarrow{P_0Q_0}}_{0}\right)+16\\&=\left|\overrightarrow{PP_0}+\overrightarrow{Q_0Q}\right|^2+16\end{aligned}$$

$\therefore$ $\underline{\left|\overrightarrow{PQ}\right|^2=\left|\overrightarrow{PP_0}+\overrightarrow{Q_0Q}\right|^2+16\ \text{は示された（証明終）}}$

$\boxed{3}$

(1) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{f(\tan x)}{\cos^2 x}dx$ において $\left\{\begin{array}{l} t=\tan x \\ dt=\dfrac{1}{\cos^2 x}dx \end{array} \quad \begin{array}{c|ccc} x & 0 & \to & \frac{\pi}{4} \\ \hline t & 0 & \to & 1 \end{array}\right\}$ と置換すると,

$$\underline{\int_0^{\frac{\pi}{4}} \frac{f(\tan x)}{\cos^2 x}dx=\int_0^1 f(t)dt} \quad \text{(証明終)}$$

(2) $f(x)=x^n$ とすると, $0 \leqq x \leqq 1$ で $f(x)$ は連続な関数.

(1)での結果を用いると,

$$\int_0^{\frac{\pi}{4}} \frac{\tan^n x}{\cos^2 x}dx=\int_0^1 t^n dt=\underline{\frac{1}{n+1}}$$

次に, $0 \leqq x \leqq \dfrac{\pi}{4}$ で, $\tan x \geqq 0$, $0<\cos^2 x \leqq 1$ だから

$$\int_0^{\frac{\pi}{4}} \frac{\tan^n x}{\cos^2 x}dx \geqq \int_0^{\frac{\pi}{4}} \tan^n x\,dx$$

(被積分関数を評価)

$$\iff \underline{\frac{1}{n+1} \geqq \int_0^{\frac{\pi}{4}} \tan^n x\,dx} \quad \text{(証明終)}$$

(3)

$$\begin{aligned} &1-\tan^2 x+\tan^4 x-\tan^6 x+\cdots+(-1)^n\tan^{2n} x \\ &=\frac{1-(-\tan^2 x)^{n+1}}{1-(-\tan^2 x)}=\cos^2 x\left\{1-(-\tan^2 x)^{n+1}\right\} \end{aligned}$$

ここで, $0 \leqq x \leqq \dfrac{\pi}{4}$ より, $\cos^2 x \neq 0$

辺々を $\cos^2 x$ で割れば

$$\underline{\frac{1-\tan^2 x+\tan^4 x-\tan^6 x+\cdots+(-1)^n\tan^{2n} x}{\cos^2 x}=1-(-1)^{n+1}\tan^{2(n+1)} x} \quad \text{(証明終)}$$

(4) (3)の結果を使い, 辺々を区間 $\left[0,\ \dfrac{\pi}{4}\right]$ で積分

$$\int_0^{\frac{\pi}{4}} \frac{1-\tan^2 x+\tan^4 x-\cdots+(-1)^n\tan^{2n} x}{\cos^2 x}dx=\int_0^{\frac{\pi}{4}}\left\{1-(-1)^{n+1}\tan^{2(n+1)} x\right\}dx$$

$$\begin{aligned} \iff &\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x}dx-\int_0^{\frac{\pi}{4}} \frac{\tan^2 x}{\cos^2 x}dx+\int_0^{\frac{\pi}{4}} \frac{\tan^4 x}{\cos^2 x}dx-\cdots+(-1)^n\int_0^{\frac{\pi}{4}} \frac{\tan^{2n} x}{\cos^2 x}dx \\ &=\frac{\pi}{4}-(-1)^{n+1}\int_0^{\frac{\pi}{4}} \tan^{2(n+1)} x\,dx \end{aligned}$$

$$\iff 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+(-1)^n\frac{1}{2n+1}=\frac{\pi}{4}+(-1)^n\int_0^{\frac{\pi}{4}} \tan^{2(n+1)} x\,dx \quad \cdots\cdots①$$

ここで, (2)を用いて,

$$0 \leqq \int_0^{\frac{\pi}{4}} \tan^{2(n+1)} x\,dx \leqq \frac{1}{2n+3}$$

と表せる.

(最右辺) $\to 0$ $(n \to \infty)$ から, はさみうちの原理より,

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{4}} \tan^{2(n+1)} x\,dx=0$$

①の両辺の極限をとれば,

$$\begin{aligned} &1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+(-1)^n\frac{1}{2n+1}+(-1)^{n+1}\frac{1}{2n+3}\cdots \\ &=\frac{\pi}{4}+(-1)^n\lim_{n\to\infty}\int_0^{\frac{\pi}{4}} \tan^{2(n+1)} x\,dx \end{aligned}$$

$$\therefore\ \underline{\lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k\frac{1}{2k+1}=\frac{\pi}{4}}$$

$\boxed{4}$

X を6以外の目とする．

(1)　・3回のとき

$(6,\ 6,\ X)$ または $(X,\ 6,\ 6)$

$$\therefore\ p_3=2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)=\frac{10}{216}=\frac{5}{108}\qquad\qquad \therefore\ \underline{p_3=\frac{5}{108}}$$

・4回のとき

$(6,\ 6,\ X,\ X)$ または $(X,\ 6,\ 6,\ X)$ または $(X,\ X,\ 6,\ 6)$

$$\therefore\ p_4=3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2=\frac{75}{1296}=\frac{25}{432}\qquad\qquad \therefore\ \underline{p_4=\frac{25}{432}}$$

(2)　n 回中に2回だけ6の目が出て，しかも連続であるのは，

$(6,\ 6,\ X,\ X,\ \cdots X)$　または $\cdots$ または　$(X,\ X,\ \cdots X,\ 6,\ 6)$ の $(n-1)$ 通り

$$\therefore\ \underline{p_n=(n-1)\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-2}}\quad (n\geqq 3)$$

ここで，

$$p_{n+1}-\frac{5}{6}p_n=n\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-1}-\underbrace{\frac{5}{6}\cdot(n-1)\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-2}}$$

$$=n\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-1}-(n-1)\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-1}$$

$$=\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-1}$$

$$\therefore\ \underline{p_{n+1}-\frac{5}{6}p_n=\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-1}\quad (n\geqq 3)}\ （証明終）$$

(3)　(2)を用いて，$6p_{k+1}-5p_k=\dfrac{1}{6}\left(\dfrac{5}{6}\right)^{k-1}$　$(k\geqq 3)$

$k=3,4,5,\cdots,n-1$ まで代入し辺々和をとる．

$$(6p_4-5p_3)+(6p_5-5p_4)+(6p_6-5p_5)+\cdots+(6p_n-5p_{n-1})$$
$$=\frac{1}{6}\left\{\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^3+\cdots+\left(\frac{5}{6}\right)^{n-2}\right\}$$

$$\Longleftrightarrow -5p_3+p_4+p_5+p_6+\cdots+p_{n-1}+6p_n=\frac{1}{6}\cdot\frac{\frac{25}{36}\left\{1-\left(\frac{5}{6}\right)^{n-3}\right\}}{1-\frac{5}{6}}$$

$$\Longleftrightarrow p_3+p_4+p_5+\cdots+p_{n-1}+p_n=\frac{25}{36}\left\{1-\left(\frac{5}{6}\right)^{n-3}\right\}+6p_3-5p_n$$

$$\Longleftrightarrow S_n=\frac{25}{36}\left\{1-\left(\frac{5}{6}\right)^{n-3}\right\}+6\cdot\frac{5}{108}-5\cdot(n-1)\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{n-2}$$

$$\lim_{n\to\infty}S_n=\frac{25}{36}+\frac{5}{18}=\underline{\frac{35}{36}}\qquad\left(\because\ \lim_{n\to\infty}\left(\frac{5}{6}\right)^{n-3}=0,\ \lim_{n\to\infty}(n-1)\left(\frac{5}{6}\right)^{n-2}=0\right)$$