$\boxed{1}$

$t=\cos\theta-\sin\theta$ のとき，

$$t^2=\cos^2\theta-2\cos\theta\sin\theta+\sin^2\theta=1-2\sin\theta\cos\theta$$

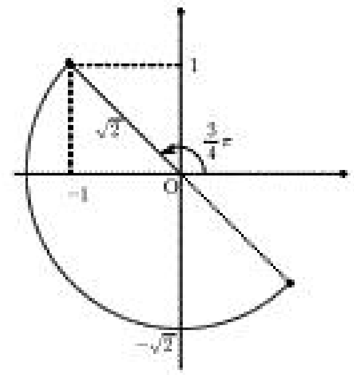
$$\therefore\ 2\sin\theta\cos\theta=1-t^2$$

(1)

$$\begin{aligned} f\ \theta\ &=2\sin2\theta+4a\ \ \cos\theta-\sin\theta\ \ +1 \\ &=2\cdot2\sin\theta\cos\theta+4a\ \ \cos\theta-\sin\theta\ \ +1 \\ &=2\ \ 1-t^2\ \ +4at+1 \\ &=-2t^2+4at+3 \end{aligned}$$

… 答

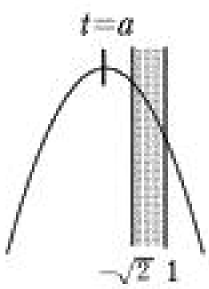
(2) $t=\cos\theta-\sin\theta=\sqrt{2}\sin\left(\theta+\dfrac{3}{4}\pi\right)$



よって，$0\leqq\theta\leqq\pi$ のとき，

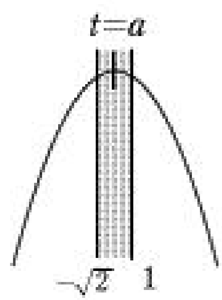
$-\sqrt{2}\leqq t\leqq1$ … 答

(3) $f\ \theta\ =-2\ \ t-a\ ^2+2a^2+3=g\ \ t$ とおく．



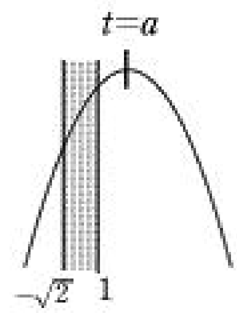
$a<-\sqrt{2}$ のとき，

最大値 $g\ \ -\sqrt{2}\ \ =-4\sqrt{2}a-1$



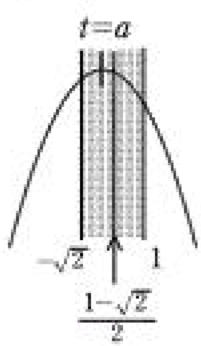
$-\sqrt{2}\leqq a\leqq1$ のとき，

最大値 $g\ \ a\ \ =2a^2+3$



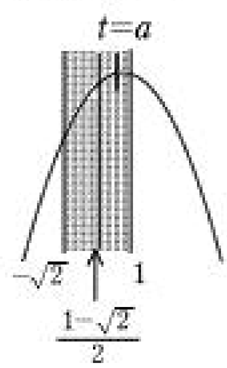
$1<a$ のとき，

最大値 $g\ \ 1\ \ =4a+1$



$a<\dfrac{1-\sqrt{2}}{2}$ のとき，

最小値 $g\ \ 1\ \ =4a+1$



$\dfrac{1-\sqrt{2}}{2}\leqq a$ のとき，

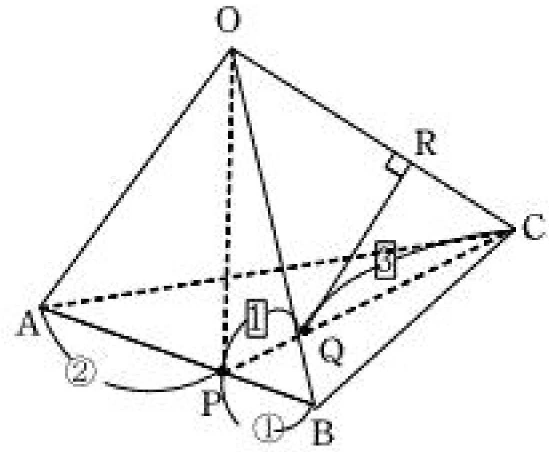
最小値 $g\ \ -\sqrt{2}\ \ =-4\sqrt{2}a-1$

以上より，$a\geqq0$ に注意して，

$$\begin{cases} 最大値\begin{cases} 2a^2+3 & 0\leqq a\leqq1 \\ 4a+1 & a>1 \end{cases} \\ \\ 最小値\ -4\sqrt{2}a-1 \end{cases}$$

…. 答

$\boxed{2}$



$$\left|\vec{a}\right|=\left|\vec{b}\right|=\left|\vec{c}\right|=1,$$

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=1\cdot1\cdot\cos60^\circ=\frac{1}{2}$$

(1)

$$\overrightarrow{OQ}=\frac{3}{4}\overrightarrow{OP}+\frac{1}{4}\overrightarrow{OC}=\frac{3}{4}\left(\frac{1}{3}\overrightarrow{OA}+\frac{2}{3}\overrightarrow{OB}\right)+\frac{1}{4}\overrightarrow{OC}$$

$$=\frac{1}{4}\vec{a}+\frac{1}{2}\vec{b}+\frac{1}{4}\vec{c} \qquad \cdots \text{答}$$

$$\left|\overrightarrow{OQ}\right|^2=\frac{1}{16}\left|\vec{a}\right|^2+\frac{1}{4}\left|\vec{b}\right|^2+\frac{1}{16}\left|\vec{c}\right|^2+\frac{1}{4}\vec{a}\cdot\vec{b}+\frac{1}{4}\vec{b}\cdot\vec{c}+\frac{1}{8}\vec{c}\cdot\vec{a}=\frac{11}{16}$$

$$\therefore \left|\overrightarrow{OQ}\right|=\frac{\sqrt{11}}{4} \qquad \cdots \text{答}$$

(2) $\overrightarrow{OR}=t\,\overrightarrow{OC}$ とおく.

$\overrightarrow{OC}\perp\overrightarrow{QR}$ より, $\overrightarrow{OC}\cdot\overrightarrow{QR}=0$　　$\vec{c}\cdot\ \overrightarrow{OR}-\overrightarrow{OQ}\ =0$

$$\vec{c}\cdot\left(t\vec{c}-\frac{1}{4}\vec{a}-\frac{1}{2}\vec{b}-\frac{1}{4}\vec{c}\right)=0$$

$$t-\frac{1}{4}\cdot\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{4}=0\ \therefore\ t=\frac{5}{8}$$

よって, $\left|\overrightarrow{OR}\right|=\left|\overrightarrow{RC}\right|=5:3$ $\qquad\cdots$ 答

(3)

$$\triangle OQR=\triangle OQC\times\frac{5}{8}=\left(\triangle OPC\times\frac{3}{4}\right)\times\frac{5}{8}=\triangle OPC\times\frac{15}{32}$$

ここで, $\left|\overrightarrow{OP}\right|^2=\left|\frac{1}{3}\vec{a}+\frac{2}{3}\vec{b}\right|^2=\frac{1}{9}\left|\vec{a}\right|^2+\frac{4}{9}\vec{a}\cdot\vec{b}+\frac{4}{9}\left|\vec{b}\right|^2=\frac{7}{9}$

$$\overrightarrow{OP}\cdot\overrightarrow{OC}=\left(\frac{1}{3}\vec{a}+\frac{2}{3}\vec{b}\right)\cdot\vec{c}=\frac{1}{3}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{2}$$

よって,

$$\triangle OQR=\triangle OPC\times\frac{15}{32}=\frac{1}{2}\sqrt{\left|\overrightarrow{OP}\right|^2\left|\overrightarrow{OC}\right|^2-\ \overrightarrow{OP}\cdot\overrightarrow{OC}\ ^2}\times\frac{15}{32}$$

$$=\frac{15}{64}\sqrt{\frac{7}{9}\cdot1-\left(\frac{1}{2}\right)^2}=\frac{15}{64}\sqrt{\frac{28-9}{36}}=\frac{15}{64}\frac{\sqrt{19}}{6}=\frac{5\sqrt{19}}{128} \qquad \cdots \text{答}$$

$\boxed{3}$ $\quad P^{-1}=\dfrac{1}{-12-2}\begin{pmatrix}-6 & -1\\ -2 & 2\end{pmatrix}=\dfrac{1}{14}\begin{pmatrix}6 & 1\\ 2 & -2\end{pmatrix}$

(1)

$$P^{-1}AP=\frac{1}{14}\begin{pmatrix}6 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}2 & 1\\ a & -3\end{pmatrix}\begin{pmatrix}2 & 1\\ 2 & -6\end{pmatrix}=\frac{1}{14}\begin{pmatrix}12+a & 3\\ 4-2a & 8\end{pmatrix}\begin{pmatrix}2 & 1\\ 2 & -6\end{pmatrix}=\frac{1}{14}\begin{pmatrix}2a+30 & a-6\\ -4a+24 & -2a-44\end{pmatrix}$$

よって，$\dfrac{2a+30}{14}=3,\ \dfrac{a-6}{14}=b,\ \dfrac{-4a+24}{14}=0,\ \dfrac{-2a-44}{14}=c$

$\therefore\ (a,b,c)=(6,0,-4)$ $\qquad$ … 答

(2) $A=\begin{pmatrix}2 & 1\\ 6 & -3\end{pmatrix}$ より，$|A|=-12\neq 0$ $\quad$ よって，A は逆行列をもつ． □

$$A^{-1}=\frac{1}{-12}\begin{pmatrix}-3 & -1\\ -6 & 2\end{pmatrix}=\frac{1}{12}\begin{pmatrix}3 & 1\\ 6 & -2\end{pmatrix} \qquad \text{… 答}$$

(3)

$$A^n=P(P^{-1}AP)^nP^{-1}=\begin{pmatrix}2 & 1\\ 2 & -6\end{pmatrix}\begin{pmatrix}3 & 0\\ 0 & -4\end{pmatrix}^n\cdot\frac{1}{14}\begin{pmatrix}6 & 1\\ 2 & -2\end{pmatrix}=\begin{pmatrix}2 & 1\\ 2 & -6\end{pmatrix}\begin{pmatrix}3^n & 0\\ 0 & (-4)^n\end{pmatrix}\cdot\frac{1}{14}\begin{pmatrix}6 & 1\\ 2 & -2\end{pmatrix}$$

$$=\frac{1}{14}\begin{pmatrix}2\cdot 3^n & (-4)^n\\ 2\cdot 3^n & -6\cdot(-4)^n\end{pmatrix}\begin{pmatrix}6 & 1\\ 2 & -2\end{pmatrix}=\frac{1}{7}\begin{pmatrix}6\cdot 3^n+(-4)^n & 3^n-(-4)^n\\ 6\cdot 3^n-6\cdot(-4)^n & 3^n+6\cdot(-4)^n\end{pmatrix} \qquad \text{… 答}$$

(4) $A+6A^{-1}=\begin{pmatrix}2 & 1\\ 6 & -3\end{pmatrix}+\dfrac{1}{2}\begin{pmatrix}3 & 1\\ 6 & -2\end{pmatrix}=\begin{pmatrix}\frac{7}{2} & \frac{3}{2}\\ 9 & -4\end{pmatrix}=B$ $\quad$ とおく．

ケーリー・ハミルトンの定理より，

$$B^2+\frac{1}{2}B-\frac{55}{2}E=O,$$

$\lambda^2+\dfrac{1}{2}\lambda-\dfrac{55}{2}=0$ を解くと，

$$\lambda=-\frac{11}{2},5$$

$B\vec{x}=-\dfrac{11}{2}\vec{x}$ となる $\vec{x}$ の1つは，$\vec{x}=\begin{pmatrix}1\\ -6\end{pmatrix}$ $\qquad$ $\therefore B^n\vec{x}=\left(-\dfrac{11}{2}\right)^n\vec{x}$

$B\vec{y}=5\vec{y}$ となる $\vec{y}$ の1つは，$\vec{y}=\begin{pmatrix}1\\ 1\end{pmatrix}$ $\qquad$ $\therefore B^n\vec{y}=5^n\vec{y}$

したがって，$B^n\begin{pmatrix}1 & 1\\ -6 & 1\end{pmatrix}=\begin{pmatrix}\left(-\frac{11}{2}\right)^n & 5^n\\ -6\cdot\left(-\frac{11}{2}\right)^n & 5^n\end{pmatrix}$

$$\therefore B^n=\begin{pmatrix}\left(-\frac{11}{2}\right)^n & 5^n\\ -6\cdot\left(-\frac{11}{2}\right)^n & 5^n\end{pmatrix}\begin{pmatrix}1 & 1\\ -6 & 1\end{pmatrix}^{-1}=\frac{1}{7}\begin{pmatrix}\left(-\frac{11}{2}\right)^n+6\cdot 5^n & -\left(-\frac{11}{2}\right)^n+5^n\\ -6\cdot\left(-\frac{11}{2}\right)^n+6\cdot 5^n & 6\cdot\left(-\frac{11}{2}\right)^n+5^n\end{pmatrix} \qquad \text{… 答}$$

$\boxed{4}$

(1) $f\ x = -4x^2+2\ e^{-x^2}$ について，

$$f'\ x = 4x\ 2x^2-3\ e^{-x^2}$$

x	$\cdots$	$-\sqrt{\frac{3}{2}}$	$\cdots$	0	$\cdots$	$\sqrt{\frac{3}{2}}$	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	↘		↗		↘		↗

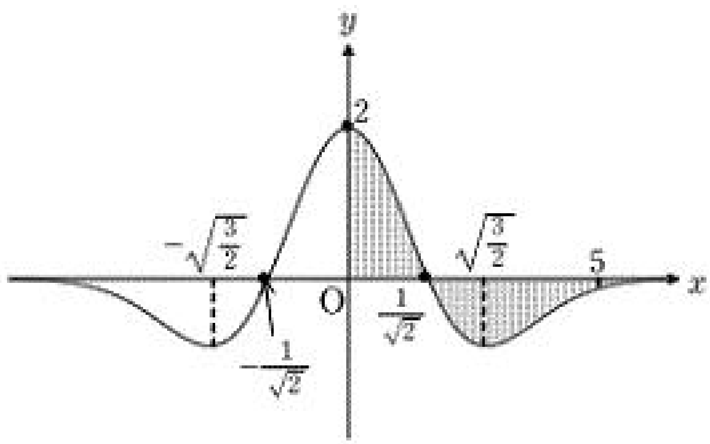
増減表より，

$$\begin{cases} \text{極大値 } f\ 0\ = 2 & \cdots \text{答} \\ \text{極小値 } f\left(\pm\sqrt{\dfrac{3}{2}}\right) = -4e^{-\frac{3}{2}} & \cdots \text{答} \end{cases}$$

(2) $$\int_0^a x^2 e^{-x^2} dx = \left[-\frac{x}{2}\cdot e^{-x^2}\right]_0^a - \int_0^a \left(-\frac{1}{2}\right)\cdot e^{-x^2} dx = -\frac{a}{2}e^{-a^2} + \frac{1}{2}I\ a \quad \cdots \text{答}$$

(3) $f\ x = f\ -x$ より，$y=f\ x$ はy軸に関して対称

また，$\lim\limits_{x\to\infty} f\ x\ = 0$ より,グラフは図のようになる．



よって,求める面積は,

$$\int_0^{\frac{1}{\sqrt{2}}} f\ x\ dx + \int_{\frac{1}{\sqrt{2}}}^5 -f\ x\ \ dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} -4x^2+2\ e^{-x^2} dx + \int_{\frac{1}{\sqrt{2}}}^5 4x^2-2\ e^{-x^2} dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} -4x^2+2\ e^{-x^2} dx + \int_0^5 4x^2-2\ e^{-x^2} dx - \int_0^{\frac{1}{\sqrt{2}}} 4x^2-2\ e^{-x^2} dx$$

$$= \int_0^5 4x^2-2\ e^{-x^2} dx - 2\int_0^{\frac{1}{\sqrt{2}}} 4x^2-2\ e^{-x^2} dx$$

$$= 4\int_0^5 x^2 e^{-x^2} dx - 2\int_0^5 e^{-x^2} dx - 8\int_0^{\frac{1}{\sqrt{2}}} x^2 e^{-x^2} dx + 4\int_0^{\frac{1}{\sqrt{2}}} e^{-x^2} dx$$

$$= 4\left\{-\frac{5}{2}e^{-25} + \frac{1}{2}I\ 5\right\} - 2I\ 5\ - 8\left\{-\frac{1}{2\sqrt{2}}e^{-\frac{1}{2}} + \frac{1}{2}I\left(\frac{1}{\sqrt{2}}\right)\right\} + 4I\left(\frac{1}{\sqrt{2}}\right)$$

$$= -10e^{-25} + 2\sqrt{2}e^{-\frac{1}{2}} \quad \cdots \text{答}$$

$\boxed{5}$

(1)

$$\left|\int_0^1 \frac{x^2}{1+x^2}dx - a_n\right| = \left|\int_0^1 \frac{x^2}{1+x^2}dx - \int_0^1 \frac{x^2+(-x^2)^{n+1}}{1+x^2}dx\right| = \left|\int_0^1 \frac{-(-x^2)^{n+1}}{1+x^2}dx\right|$$

$$\leqq \int_0^1 \left|\frac{-(-x^2)^{n+1}}{1+x^2}\right|dx = \int_0^1 \frac{x^{2n+2}}{1+x^2}dx \leqq \int_0^1 \frac{x^{2n+2}}{1+0^2}dx$$

$$= \int_0^1 x^{2n+2}dx = \left[\frac{1}{2n+3}x^{2n+3}\right]_0^1 = \frac{1}{2n+3} \quad \square$$

(2) $\dfrac{x^2}{1+x^2} = \dfrac{1+x^2-1}{1+x^2} = 1 - \dfrac{1}{1+x^2}$

$\displaystyle\int_0^1 \frac{1}{1+x^2}dx$ で $x=\tan\theta$ とおくと，

$$dx = \frac{1}{\cos^2\theta}d\theta$$

x	$0 \to 1$
θ	$0 \to \frac{\pi}{4}$

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \int_0^{\frac{\pi}{4}} \frac{1}{1+\tan^2\theta}\cdot\frac{1}{\cos^2\theta}d\theta = \int_0^{\frac{\pi}{4}} d\theta = \left[\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$

よって，

$$\int_0^1 \frac{x^2}{1+x^2}dx = \int_0^1 \left(1 - \frac{1}{1+x^2}\right)dx = \int_0^1 1dx - \int_0^1 \frac{1}{1+x^2}dx = \left[x\right]_0^1 - \frac{\pi}{4} = 1 - \frac{\pi}{4} \quad \cdots \text{答}$$

(3) $\dfrac{x^2+(-x^2)^{n+1}}{1+x^2} = x^2\cdot\dfrac{1-(-x^2)^n}{1-(-x^2)}$　これは,初項 x^2 ,公比 $-x^2$ の等比数列の第 n 項までの和

$$\therefore a_n = \int_0^1 \frac{x^2+(-x^2)^{n+1}}{1+x^2}dx = \int_0^1 \sum_{k=1}^{n} x^2\cdot(-x^2)^{k-1}dx = \int_0^1 \sum_{k=1}^{n}(-1)^{k-1}x^{2k}dx = \sum_{k=1}^{n}\int_0^1 (-1)^{k-1}x^{2k}dx$$

$$= \sum_{k=1}^{n}\left[\frac{(-1)^{k-1}}{2k+1}x^{2k+1}\right]_0^1 = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k+1} = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{2k+1} \quad \square$$

(4) $0 \leqq \left|\displaystyle\int_0^1 \frac{x^2}{1+x^2}dx - a_n\right| \leqq \dfrac{1}{2n+3}$ であり， $\displaystyle\lim_{n\to\infty}\frac{1}{2n+3} = 0$ であるから，はさみうちの原理より，

$$\lim_{n\to\infty}\left|\int_0^1 \frac{x^2}{1+x^2}dx - a_n\right| = 0$$

$$\therefore \lim_{n\to\infty} a_n = \int_0^1 \frac{x^2}{1+x^2}dx = 1 - \frac{\pi}{4} \qquad \therefore \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{2k+1} = 1 - \frac{\pi}{4} \quad \cdots \text{答}$$