[I]

$$\int_{-1}^{1} f(x)dx = 2\int_0^1 (3ax^2+b)dx = 2a+2b = 0 \quad (a+b=0 \quad \cdots\cdots①)$$

$$\int_{-1}^{1}\{f(x)\}^2 dx = 2\int_0^1 \{9a^2x^4+(6ab+4)x^2+b^2\}dx = \frac{18}{5}a^2+\frac{2}{3}(6ab+4)+2b^2=6$$

$$(27a^2+30ab+15b^2=25 \quad \cdots\cdots②)$$

①，②より，　$27a^2-30a^2+15a^2=25$

$$\iff a^2=\frac{25}{12} \qquad \therefore\ a=\frac{\overset{(ア)}{\boxed{5}}\sqrt{\overset{(イ)}{\boxed{3}}}}{\underset{(ウ)}{\boxed{6}}}\ (>0) \qquad \cdots(答)$$

$$\therefore\ b=-\frac{\overset{(エ)}{\boxed{5}}\sqrt{\overset{(オ)}{\boxed{3}}}}{\underset{(カ)}{\boxed{6}}} \qquad \cdots(答)$$

$$\int_0^1 \cos\left(\frac{\pi x}{2}\right)dx = \left[\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_0^1 = \frac{\overset{(キ)}{\boxed{2}}}{\pi} \qquad \cdots(答)$$

$$\int_0^1 \cos^2\left(\frac{\pi x}{2}\right)dx = \int_0^1 \frac{1+\cos\pi x}{2}dx = \frac{1}{2}\left[x+\frac{1}{\pi}\sin\pi x\right]_0^1 = \frac{\overset{(ク)}{\boxed{1}}}{\underset{(ケ)}{\boxed{2}}} \qquad \cdots(答)$$

$$\begin{aligned}
\int_0^1 x^2\cos\left(\frac{\pi x}{2}\right)dx &= \int_0^1 x^2\left\{\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right\}' dx \\
&= \left[\frac{2}{\pi}x^2\sin\left(\frac{\pi x}{2}\right)\right]_0^1 - \int_0^1 \frac{4}{\pi}x\sin\left(\frac{\pi x}{2}\right)dx \\
&= \frac{2}{\pi}+\frac{4}{\pi}\int_0^1 x\left\{\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right\}' dx \\
&= \frac{2}{\pi}+\frac{8}{\pi^2}\left[x\cos\left(\frac{\pi x}{2}\right)\right]_0^1 - \frac{8}{\pi^2}\int_0^1 \cos\left(\frac{\pi x}{2}\right)dx \\
&= \frac{2}{\pi}-\frac{8}{\pi^2}\left[\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_0^1 \\
&= \frac{\overset{(コ)}{\boxed{2}}}{\pi}-\frac{\overset{(サ)}{\boxed{16}}}{\pi^{\underset{(シ)}{\boxed{3}}}} \qquad \cdots(答)
\end{aligned}$$

$$\begin{aligned}
F &= \int_{-1}^{1}\left\{\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)-(pf(x)+q)\right\}^2 dx \\
&= \int_{-1}^{1}\left[\frac{\pi^2}{4}\cos^2\left(\frac{\pi x}{2}\right)+p^2\{f(x)\}^2+q^2-\pi pf(x)\cos\left(\frac{\pi x}{2}\right)+2pqf(x)-\pi q\cos\left(\frac{\pi x}{2}\right)\right]dx \\
&= \frac{\pi^2}{4}+6p^2+2q^2-\pi p\cdot 6a\left(\frac{2}{\pi}-\frac{16}{\pi^3}\right)-\pi p\cdot 2b\cdot\frac{2}{\pi}-2\pi q\cdot\frac{2}{\pi} \\
&= \frac{\pi^2}{4}+6p^2+2q^2-5\sqrt{3}\pi p\left(\frac{2}{\pi}-\frac{16}{\pi^3}\right)+\frac{10\sqrt{3}}{3}p-4q \\
&= 6\left\{p-\left(\frac{5\sqrt{3}}{9}-\frac{20\sqrt{3}}{3\pi^2}\right)\right\}^2+2(q-1)^2+\underbrace{\frac{\pi^2}{4}-2-6\left(\frac{5\sqrt{3}}{9}-\frac{20\sqrt{3}}{3\pi^2}\right)^2}_{(定数)}
\end{aligned}$$

$$\left(\begin{aligned} a&=\frac{5\sqrt{3}}{6} \\ b&=-\frac{5\sqrt{3}}{6}\end{aligned}\right)$$

これより，

$$p=\frac{\overset{(ス)}{\boxed{5}}\sqrt{\overset{(セ)}{\boxed{3}}}}{\underset{(ソ)}{\boxed{9}}}-\frac{\overset{(タ)}{\boxed{20}}\sqrt{\overset{(チ)}{\boxed{3}}}}{\underset{(ツ)}{\boxed{3}}\pi^{\underset{(テ)}{\boxed{2}}}},\ q=\overset{(ト)}{\boxed{1}}$$ のとき F は最小となる． $\cdots$（答）

[II]

問１　$\overrightarrow{n}=\left(-2, \sqrt{5}, \sqrt{3}\right)$に垂直な点$\left(1, -\dfrac{\sqrt{5}}{2}, \dfrac{3\sqrt{3}}{2}\right)$を通る平面方程式は，

$$-2(x-1)+\sqrt{5}\left(y+\frac{\sqrt{5}}{2}\right)+\sqrt{3}\left(z-\frac{3\sqrt{3}}{2}\right)=0$$

$$\Longleftrightarrow\ -2x+\sqrt{5}y+\sqrt{3}z=0\quad \therefore\ (\text{平面 } \alpha)\ 2x-\sqrt{5}y-\sqrt{3}z=0 \qquad \cdots(\text{答})$$

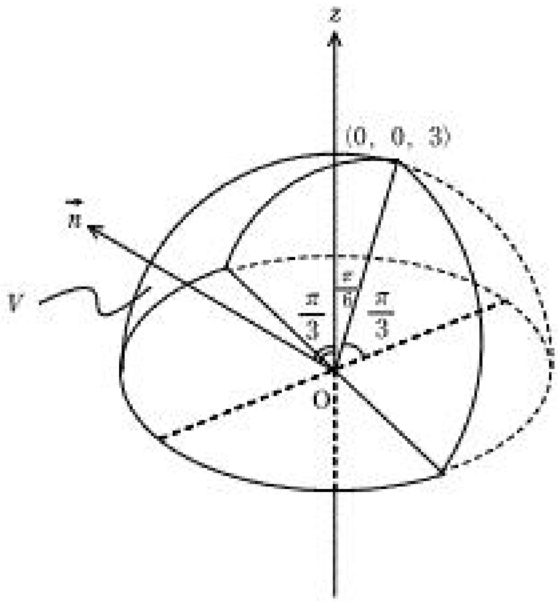
問２　$\overrightarrow{e_3}=(0, 0, 1)$とすると，$\left|\overrightarrow{e_3}\right|=1$　さらに，$\left|\overrightarrow{n}\right|=\sqrt{4+5+3}=2\sqrt{3}$

$\overrightarrow{e_3}$ と $\overrightarrow{n}$ のなす角をθ $(0\leqq\theta\leqq\pi)$より，

$$\cos\theta=\frac{\overrightarrow{e_3}\cdot\overrightarrow{n}}{\left|\overrightarrow{e_3}\right|\left|\overrightarrow{n}\right|}=\frac{\sqrt{3}}{1\cdot 2\sqrt{3}}=\frac{1}{2} \qquad \therefore\ \theta=\frac{\pi}{3} \qquad \cdots(\text{答})$$

問３　求める体積Vは半径3の半球体の$\dfrac{2}{3}$の体積．

$$V=\frac{1}{2}\cdot\left(\frac{4}{3}\pi\cdot 3^3\right)\cdot\frac{2}{3}$$
$$=12\pi \qquad \cdots(\text{答})$$



[Ⅲ]

問 1

$$\begin{cases} x=\dfrac{k}{1+t^2k^2} \\ y=\dfrac{tk^2}{1+t^2k^2} \end{cases} \quad (t>0,\ k>0) \quad \text{より，}\ \frac{y}{x}=tk \quad \left(t=\frac{y}{kx}\right)$$

これより，

$$x=\frac{k}{1+\left(\dfrac{y}{kx}\right)^2k^2} \iff x(x^2+y^2)=kx^2 \quad \cdots\cdots①$$

ここで，$x=\dfrac{k}{1+t^2k^2}>0$，$y=\dfrac{tk^2}{1+t^2k^2}>0$ より，

$$\begin{aligned} ① &\iff x^2+y^2-kx=0 \\ &\iff \left(x-\frac{k}{2}\right)^2+y^2=\left(\frac{k}{2}\right)^2 \end{aligned}$$

∴ $C(k)$は点$\left(\overset{(ア)}{\boxed{\dfrac{k}{2}}},\ \overset{(イ)}{\boxed{0}}\right)$を中心とする，半径$\overset{(ウ)}{\boxed{\dfrac{k}{2}}}$の円の一部である． …（答）

問 2　$\left(k=\dfrac{1}{2}\right) \to T_t\left(\dfrac{1}{2}\right)=\left(\dfrac{2}{t^2+4},\ \dfrac{t}{t^2+4}\right)$　円の中心は$\left(\dfrac{1}{4},\ 0\right)$より，

$$l_1(t):y=\frac{\dfrac{t}{t^2+4}}{\dfrac{2}{t^2+4}-\dfrac{1}{4}}\left(x-\frac{1}{4}\right) \iff 4tx-(4-t^2)y-t=0 \quad \cdots\text{（答）}$$

$(k=1) \to T_t(1)=\left(\dfrac{1}{t^2+1},\ \dfrac{t}{t^2+1}\right)$　円の中心は$\left(\dfrac{1}{2},\ 0\right)$より，

$$l_2(t):y=\frac{\dfrac{t}{t^2+1}}{\dfrac{1}{t^2+1}-\dfrac{1}{2}}\left(x-\frac{1}{2}\right) \iff 2tx-(1-t^2)y-t=0 \quad \cdots\text{（答）}$$

問 3

$$\begin{cases} 4tx-(4-t^2)y=t \\ 4tx-2(1-t^2)y=2t \end{cases} \qquad \therefore\ P(t)=\left(\frac{3}{2(t^2+2)},\ \frac{t}{t^2+2}\right) \quad \cdots\text{（答）}$$

問 4　問 3 の結果より，

$$\begin{cases} x=\dfrac{3}{2(t^2+2)} \\ y=\dfrac{t}{t^2+2} \end{cases} \quad (t>0)$$

より，

$$\frac{y}{x}=\frac{2t}{3} \quad \left(t=\frac{3y}{2x}\right)$$

これより，

$$\begin{aligned} x&=\frac{3}{2\left(\dfrac{9y^2}{4x^2}+2\right)} \\ &\iff x(8x^2+9y^2)=6x^2 \quad \cdots\cdots② \end{aligned}$$

ここで，

$$x=\frac{3}{2(t^2+2)}>0,\ y=\frac{t}{t^2+2}>0$$

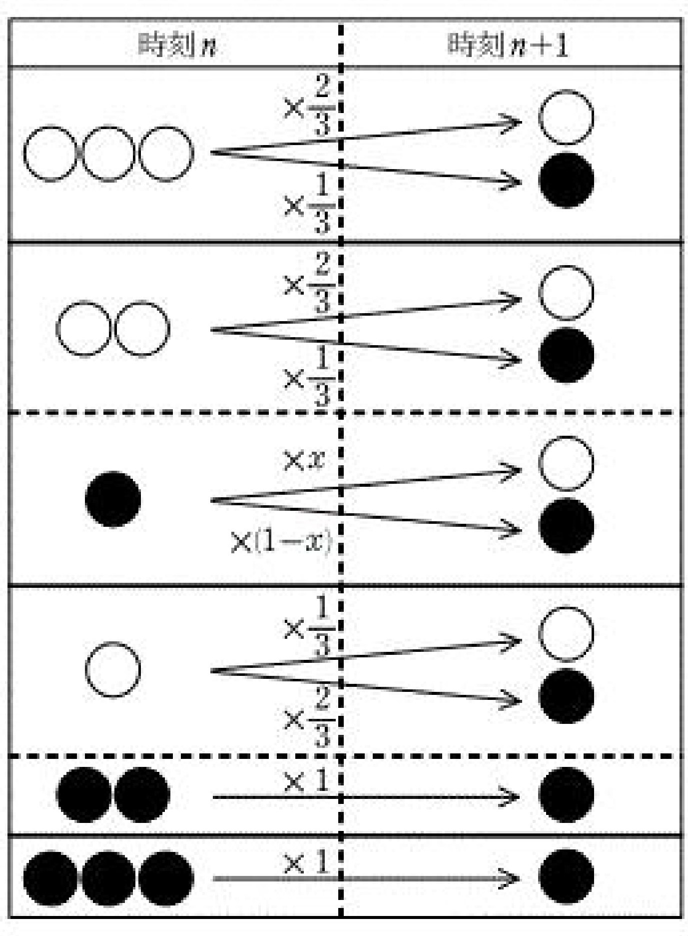
より，

$$\begin{aligned} ② &\iff 8x^2+9y^2=6x \\ &\iff \frac{\left(x-\dfrac{3}{8}\right)^2}{\dfrac{9}{64}}+\frac{y^2}{\dfrac{1}{8}}=1 \end{aligned}$$

これより，点$p(t)$の描く曲線は楕円の一部となり，

焦点の座標$\left(\dfrac{1}{4},\ 0\right)$，$\left(\dfrac{1}{2},\ 0\right)$，（長軸の長さ）$=\dfrac{3}{4}$，（短軸の長さ）$=\dfrac{1}{\sqrt{2}}$ …（答）

[IV]



$w(n)=3$ ……p_n

$w(n)=2$ ……q_n

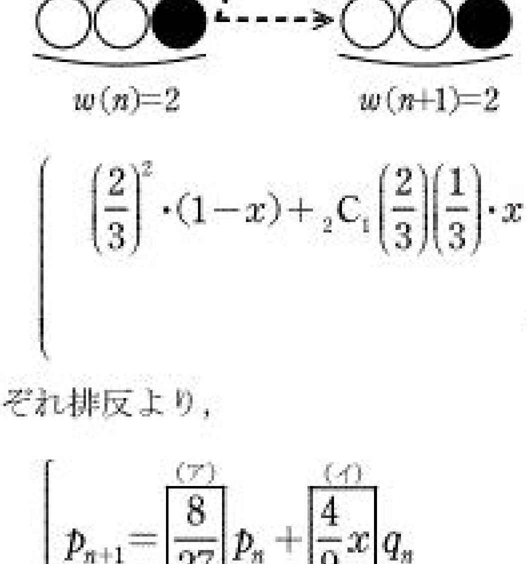
$w(n)=1$ ……r_n

問 1

$$p_1=\left(\frac{2}{3}\right)^3=\frac{8}{27} \qquad \cdots（答）$$

$$q_1={}_3\mathrm{C}_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1=\frac{12}{27}=\frac{4}{9} \qquad \cdots（答）$$

問 2　推移図を描くと下図の通り．



$$\left(\begin{aligned}\left(\frac{2}{3}\right)^2\cdot(1-x)+{}_2\mathrm{C}_1\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\cdot x&=\frac{4}{9}(1-x)+\frac{4}{9}x\\&=\frac{4}{9}\end{aligned}\right)$$

それぞれ排反より，

$$\begin{cases}p_{n+1}=\overset{(ア)}{\boxed{\dfrac{8}{27}}}p_n+\overset{(イ)}{\boxed{\dfrac{4}{9}x}}q_n\\ q_{n+1}=\underset{(ウ)}{\boxed{\dfrac{4}{9}}}p_n+\underset{(エ)}{\boxed{\dfrac{4}{9}}}q_n\end{cases} \qquad \cdots\cdots① \quad \cdots（答）$$

$$(n=0,1,2,3,\cdots\cdots)$$

問 3　①より，$243p_{n+2}-180p_{n+1}+(32-48x)p_n=0$　……②

ここで，$t=\alpha,\beta$ $(\alpha<\beta)$を2解にもつtの2次方程式

$$243t^2-180t+(32-48x)=0$$

を解くと，$t=\dfrac{10\pm2\sqrt{36x+1}}{27}$

より，

$$\alpha=\frac{10-2\sqrt{36x+1}}{27},\ \beta=\frac{10+2\sqrt{36x+1}}{27} \quad \cdots（答）$$

問 4

$$② \iff \begin{cases}p_{n+2}-\alpha p_{n+1}=\beta(p_{n+1}-\alpha p_n)\\ p_{n+2}-\beta p_{n+1}=\alpha(p_{n+1}-\beta p_n)\end{cases}$$

$$\therefore \begin{cases}p_{n+1}-\alpha p_n=(p_1-\alpha p_o)\cdot\beta^n\\ p_{n+1}-\beta p_n=(p_1-\beta p_o)\cdot\alpha^n\end{cases}$$

ここで，$p_o=1,\ p_1=\dfrac{8}{27},\ \beta-\alpha=\dfrac{4\sqrt{36x+1}}{27}$

より，

$$\frac{4\sqrt{36x+1}}{27}p_n=\left(\frac{8}{27}-\frac{10-2\sqrt{36x+1}}{27}\right)\cdot\beta^n-\left(\frac{8}{27}-\frac{10+2\sqrt{36x+1}}{27}\right)\cdot\alpha^n$$

$$\begin{aligned}\iff p_n&=\frac{-2+2\sqrt{36x+1}}{4\sqrt{36x+1}}\beta^n+\frac{2+2\sqrt{36x+1}}{4\sqrt{36x+1}}\alpha^n\\&=\underbrace{\frac{\sqrt{36x+1}+1}{2\sqrt{36x+1}}}_{F(x)}\alpha^n+\underbrace{\frac{\sqrt{36x+1}-1}{2\sqrt{36x+1}}}_{G(x)}\beta^n\end{aligned}$$

同様に，

$$\begin{cases}q_{n+1}-\alpha q_n=(q_1-\alpha q_o)\cdot\beta^n\\ q_{n+1}-\beta q_n=(q_1-\beta q_o)\cdot\alpha^n\end{cases}$$

より求めると，

$$q_n=\underbrace{-\frac{3}{\sqrt{36x+1}}}_{H(x)}\alpha^n+\underbrace{\frac{3}{\sqrt{36x+1}}}_{I(x)}\beta^n$$

以上より，

$$F(x)=\frac{\sqrt{36x+1}+1}{2\sqrt{36x+1}},\ G(x)=\frac{\sqrt{36x+1}-1}{2\sqrt{36x+1}},\ H(x)=-\frac{3}{\sqrt{36x+1}},\ I(x)=\frac{3}{\sqrt{36x+1}}$$

…（答）

問 5

r_{n+1}をr_nを用いて表すと，

$$\begin{aligned}r_{n+1}&=\frac{1}{3}r_n+\frac{2}{9}p_n+\frac{4-3x}{9}q_n\\&=\frac{1}{3}r_n+\frac{2}{9}\cdot\{F(x)\cdot\alpha^n+G(x)\cdot\beta^n\}+\frac{4-3x}{9}\cdot\{H(x)\cdot\alpha^n+I(x)\cdot\beta^n\}\\&=\frac{1}{3}r_n+\Big\{\underbrace{\frac{2}{9}F(x)+\frac{4-3x}{9}H(x)}_{A}\Big\}\alpha^n+\Big\{\underbrace{\frac{2}{9}G(x)+\frac{4-3x}{9}I(x)}_{B}\Big\}\beta^n\\&=\frac{1}{3}r_n+A\cdot\alpha^n+B\cdot\beta^n \quad (\text{上記のように}A,B\text{を定める})\end{aligned}$$

$$\iff r_{n+1}+\frac{3A}{1-3\alpha}\alpha^{n+1}+\frac{3B}{1-3\beta}\beta^{n+1}=\frac{1}{3}\left(r_n+\frac{3A}{1-3\alpha}\alpha^n+\frac{3B}{1-3\beta}\beta^n\right)$$

$$\therefore\ r_n+\frac{3A}{1-3\alpha}\alpha^n+\frac{3B}{1-3\beta}\beta^n=\left(0+\frac{3A}{1-3\alpha}+\frac{3B}{1-3\beta}\right)\left(\frac{1}{3}\right)^n$$

$$\iff r_n=\left(\frac{3A}{1-3\alpha}+\frac{3B}{1-3\beta}\right)\left(\frac{1}{3}\right)^n-\frac{3A}{1-3\alpha}\alpha^n-\frac{3B}{1-3\beta}\beta^n$$

これより，

$$\frac{r_n}{q_n}=\frac{\left(\frac{3A}{1-3\alpha}+\frac{3B}{1-3\beta}\right)\left(\frac{1}{3}\right)^n-\frac{3A}{1-3\alpha}\alpha^n-\frac{3B}{1-3\beta}\beta^n}{H(x)\cdot\alpha^n+I(x)\cdot\beta^n}$$

$$\to-\frac{3B}{(1-3\beta)I(x)}\ (n\to\infty) \quad \left(\because\ \beta=\frac{10+2\sqrt{36x+1}}{27}>\frac{10+2}{27}>\frac{1}{3}\right)$$

よって，$\displaystyle\lim_{n\to\infty}\frac{r_n}{q_n}$が存在し，かつ$\displaystyle\lim_{n\to\infty}\frac{r_n}{q_n}<1$が成立するための必要十分条件は，

$$\begin{aligned}-\frac{3B}{(1-3\beta)I(x)}&=-\frac{\frac{2}{3}\cdot\frac{\sqrt{36x+1}-1}{2\sqrt{36x+1}}+\frac{4-3x}{3}\cdot\frac{3}{\sqrt{36x+1}}}{\left(1-3\cdot\frac{10+2\sqrt{36x+1}}{27}\right)\cdot\frac{3}{\sqrt{36x+1}}}\\&=\frac{-\sqrt{36x+1}+1-3(4-3x)}{9\left(1-\frac{10+2\sqrt{36x+1}}{9}\right)}\\&=\frac{9x-11-\sqrt{36x+1}}{-1-2\sqrt{36x+1}}<1\end{aligned}$$

$$\begin{aligned}(\text{与式})&\iff\sqrt{36x+1}>10-9x\ (>0)\\&\therefore\ 36x+1>(10-9x)^2\\&\iff 81x^2-216x+99<0\\&\iff 9x^2-24x+11<0\\&\therefore\ \frac{4-\sqrt{5}}{3}<x<\frac{4+\sqrt{5}}{3}\end{aligned}$$

ここで，$0<x<1$より，

$$\frac{4-\sqrt{5}}{3}<x<1 \qquad \cdots（答）$$