［I］

問1

$$\mathrm{P}(a, b, c) \xrightarrow[\substack{\mathrm{Y}\cdots\cdots\begin{cases}(\text{表})\,3\text{回}\\(\text{裏})\,3\text{回}\end{cases}\\ \mathrm{Z}\cdots\cdots\begin{cases}(\text{表})\,4\text{回}\\(\text{裏})\,2\text{回}\end{cases}}]{\mathrm{X}\cdots\cdots\begin{cases}(\text{表})\,2\text{回}\\(\text{裏})\,4\text{回}\end{cases}} \mathrm{P}(a-2, b, c+2)\text{へ移動する.}$$

ここで，$\mathrm{P} \equiv \mathrm{O}(0, 0, 0)$ より，

$$\begin{cases} a-2=0 \\ b=0 \\ c+2=0 \end{cases} \qquad \therefore\ a=2, b=0, c=-2 \qquad (\text{答})$$

問2　$\mathrm{A}(2, 0, -2)$ と考える．

$$\mathrm{X}\cdots\cdots\begin{cases}(\text{表})\ l\text{ 回}\\(\text{裏})\ 5-l\text{ 回}\end{cases} \qquad \mathrm{Y}\cdots\cdots\begin{cases}(\text{表})\ m\text{ 回}\\(\text{裏})\ 5-m\text{ 回}\end{cases} \qquad \mathrm{Z}\cdots\cdots\begin{cases}(\text{表})\ n\text{ 回}\\(\text{裏})\ 5-n\text{ 回}\end{cases}$$

とおくと，

$$\begin{cases} \mathrm{P}_x = 2+l-(5-l) = 2l-3 \\ \mathrm{P}_y = m-(5-m) = 2m-5 \\ \mathrm{P}_z = -2+n-(5-n) = 2n-7 \end{cases}$$

より，$\mathrm{P}(2l-3, 2m-5, 2n-7)$ へ移動する．

(1)　$\mathrm{P}_z = 2n-7 = -1 \iff n=3$

$$\therefore\ {}_5\mathrm{C}_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{10}{32} = \frac{5}{16} \qquad (\text{答})$$

(2)

$$\begin{cases} \mathrm{P}_y = 2m-5 \leqq 4 \\ \mathrm{P}_z = -1 \end{cases} \iff \begin{cases} m \leqq \dfrac{9}{2} \quad (m=0, 1, 2, 3, 4) \\ n=3 \end{cases}$$

$$\therefore\ \left\{1-\left(\frac{1}{2}\right)^5\right\}\cdot\frac{5}{16} = \frac{31}{32}\cdot\frac{5}{16} = \frac{155}{512} \qquad (\text{答})$$

(3)

$$\begin{cases} \mathrm{P}_x = 2l-3 > 2 \\ \mathrm{P}_y + \mathrm{P}_z = 2m+2n-12 = 2 \end{cases} \iff \begin{cases} l > \dfrac{5}{2} \quad (l=3, 4, 5) \\ m+n=7 \end{cases}$$

$m+n=7$ は $(m, n) = (2, 5), (3, 4), (4, 3), (5, 2)$ より，

$$\therefore\ ({}_5\mathrm{C}_3 + {}_5\mathrm{C}_4 + {}_5\mathrm{C}_5)\left(\frac{1}{2}\right)^5 \cdot ({}_5\mathrm{C}_2 \cdot {}_5\mathrm{C}_5 + {}_5\mathrm{C}_3 \cdot {}_5\mathrm{C}_4)\cdot 2\cdot\left(\frac{1}{2}\right)^{10}$$

$$= \frac{16}{32}\cdot\frac{120}{1024} = \frac{15}{256} \qquad (\text{答})$$

［Ⅱ］

問１　$L : y = 5ax + a^4$ が原点 $\mathrm{O}(0, 0)$ を通るから，

$$0 = 5a \cdot 0 + a^4 \qquad \therefore\ a = 0 \qquad \text{（答）}$$

問２

$$\text{曲線}\ C : y = |x|(6-x) + x = \begin{cases} -x^2 + 7x = -x(x-7) & (x \geqq 0) \\ x^2 - 5x = x(x-5) & (x \leqq 0) \end{cases}$$

$x \geqq 0$ において，$y = -x^2 + 7x$ より，$y' = -2x + 7$

点(1，6)で接線の傾きは　5

求める接線の方程式は

$$y = 5(x-1) + 6 \qquad \therefore\ y = 5x + 1 \qquad \text{（答）}$$

問３　$x \geqq 0$ において，$y = -x^2 + 7x$ と $y = 5ax + a^4$ を連立して，

$$-x^2 + 7x = 5ax + a^4$$

$$\Leftrightarrow \quad x^2 + (5a-7)x + a^4 = 0$$

$$D = (5a-7)^2 - 4a^4$$

$$= (2a^2 + 5a - 7)(-2a^2 + 5a - 7) \quad (\because -2a^2 + 5a - 7 < 0)$$

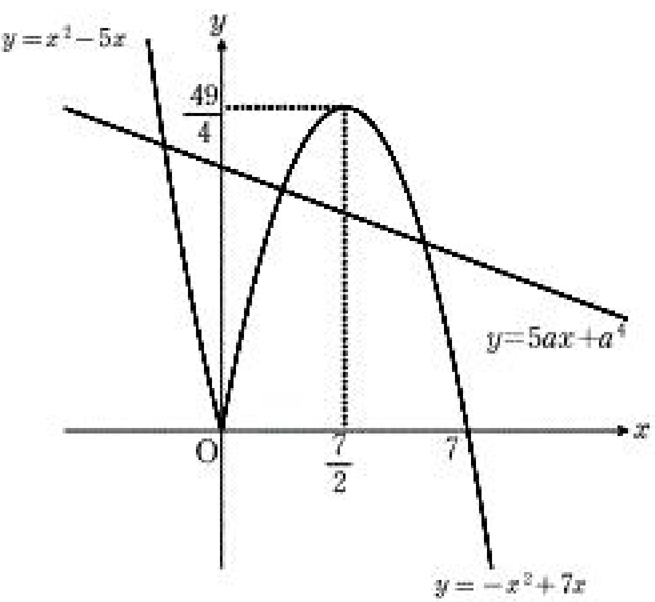
ここで，$2a^2 + 5a - 7 = (2a+7)(a-1) = 0$ とおくと，

$$a = -\frac{7}{2}, 1$$

曲線 C と直線 L が接するときの $a = -\dfrac{7}{2}, 1$ と 問１の

$a=0$ のとき $L : y=0$ に注意すれば，

$$N(a) = \begin{cases} 1 & \left(a < -\dfrac{7}{2}, 1 < a\right) \\ 2 & \left(a = -\dfrac{7}{2}, 0, 1\right) \\ 3 & \left(-\dfrac{7}{2} < a < 0, 0 < a < 1\right) \end{cases} \qquad \text{（答）}$$



[Ⅲ]

問1　2点$A\left(\dfrac{\alpha^2}{4p},\alpha\right)$, $B\left(\dfrac{\beta^2}{4p},\beta\right)$　(ただし, $\alpha<0<\beta$)を通る直線の方程式は,

$$(\alpha+\beta)y=4px+\alpha\beta$$

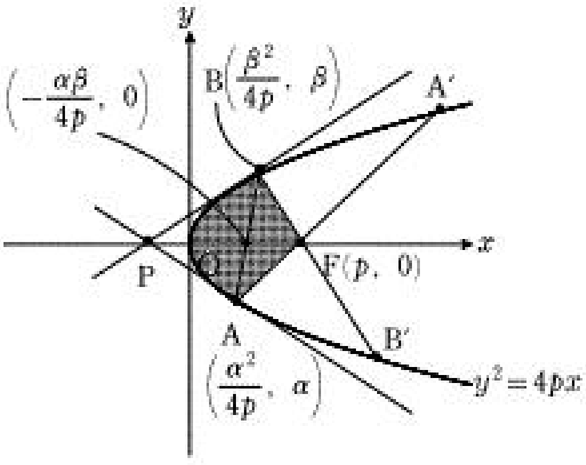
直線ABは点$\left(-\dfrac{\alpha\beta}{4p},0\right)$を通る.

(i)　$-\dfrac{\alpha\beta}{4p}\leqq p$　$(\alpha\beta\geqq-4p^2)$のとき

右図の通り2点A', B'を定める.

A', B'のy成分をA'_y, B'_yで表すと,

$$A'_y=-\frac{4p^2}{\alpha},\ B'_y=-\frac{4p^2}{\beta}$$



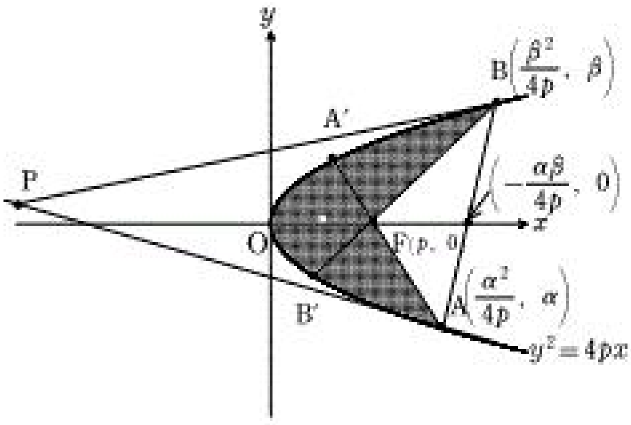
$$S=-\frac{1}{4p}\int_\alpha^\beta(y-\alpha)(y-\beta)dy+\frac{1}{2}\left(p+\frac{\alpha\beta}{4p}\right)(\beta-\alpha)$$
$$=\frac{1}{24p}(\beta-\alpha)^3+\frac{4p^2+\alpha\beta}{8p}(\beta-\alpha)\ \ \cdots\cdots①$$

(ii)　$p\leqq-\dfrac{\alpha\beta}{4p}$　$(\alpha\beta\leqq-4p^2)$のとき

$$S=-\frac{1}{4p}\int_\alpha^\beta(y-\alpha)(y-\beta)dy-\frac{1}{2}\left(-\frac{\alpha\beta}{4p}-p\right)(\beta-\alpha)$$
$$=\frac{1}{24p}(\beta-\alpha)^3+\frac{4p^2+\alpha\beta}{8p}(\beta-\alpha)\ \ \cdots\cdots②$$

以上(i), (ii)より, ①と②から,

$$S=\frac{1}{24p}(\beta-\alpha)^3+\frac{4p^2+\alpha\beta}{8p}(\beta-\alpha)\qquad(答)$$



(注意) 問題文中にある「Cと<u>2直線AF, BF</u>で囲まれる部分の面積をS」の部分を
<u>2線分AF, BF</u>　と解釈して解いてある.

問2　点Pのy成分は$\dfrac{\alpha+\beta}{2}$より,

$$T=\int_\alpha^{\frac{\alpha+\beta}{2}}\frac{1}{4p}(y-\alpha)^2dy+\int_{\frac{\alpha+\beta}{2}}^\beta\frac{1}{4p}(y-\beta)^2dy$$
$$=\frac{1}{12p}\Big[(y-\alpha)^3\Big]_\alpha^{\frac{\alpha+\beta}{2}}+\frac{1}{12p}\Big[(y-\beta)^3\Big]_{\frac{\alpha+\beta}{2}}^\beta$$
$$=\frac{1}{48p}(\beta-\alpha)^3\qquad(答)$$

問3

$$S=T$$
$$\Leftrightarrow\ \frac{1}{24p}(\beta-\alpha)^3+\frac{4p^3+\alpha\beta}{8p}(\beta-\alpha)=\frac{1}{48p}(\beta-\alpha)^3$$
$$\Leftrightarrow\ \frac{1}{48p}(\beta-\alpha)^2+\frac{4p^2+\alpha\beta}{8p}=0\qquad(\because\ \beta-\alpha>0)$$
$$\Leftrightarrow\ \beta^2+4\alpha\beta+\alpha^2+24p^2=0$$
$$\Leftrightarrow\ \left(\frac{\beta}{\alpha}\right)^2+\frac{4\beta}{\alpha}+1+\frac{24p^2}{\alpha^2}=0$$

$\left(x=-\dfrac{\beta}{\alpha}とおく\right)$

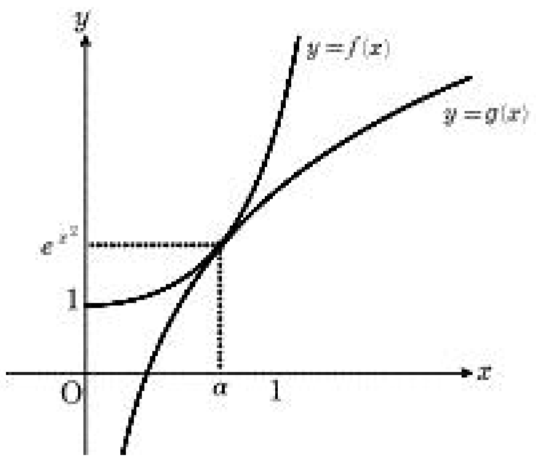
$$\Leftrightarrow\ \frac{24p^2}{\alpha^2}=-x^2+4x-1\ \ (>0)$$

上記より,

$$x^2-4x+1<0$$
$$\therefore\ 2-\sqrt{3}<x<2+\sqrt{3}$$
$$\therefore\ 2-\sqrt{3}<-\frac{\beta}{\alpha}<2+\sqrt{3}\qquad(答)$$

[IV]

問１　$f(x)=e^{x^2}\ (x>0),\ g(x)=a\log x+b\ (x>0)$ とおく．



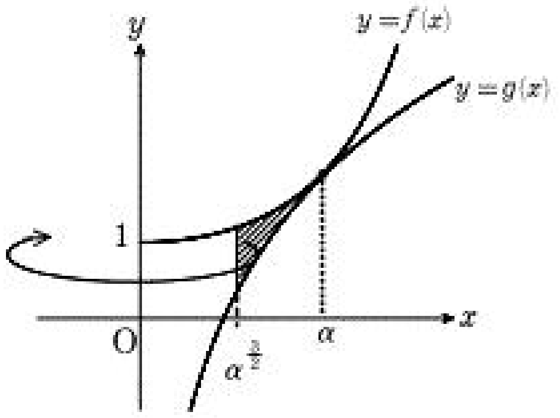
$$f'(x)=2xe^{x^2},\ g'(x)=\frac{a}{x}$$

条件より，

$$\begin{cases} f(\alpha)=g(\alpha) \\ f'(\alpha)=g'(\alpha) \end{cases} \Leftrightarrow \begin{cases} e^{\alpha^2}=a\log\alpha+b \\ 2\alpha e^{\alpha^2}=\dfrac{a}{\alpha} \end{cases}$$

$$\therefore\ a=2\alpha^2e^{\alpha^2},\ b=e^{\alpha^2}\left(1-2\alpha^2\log\alpha\right) \qquad \text{(答)}$$

問２



$$\begin{aligned} V(\alpha)&=\int_{\alpha^{\frac{3}{2}}}^{\alpha}2\pi x\{f(x)-g(x)\}dx \ \leftarrow \text{(バームクーヘン求積)} \\ &=2\pi\int_{\alpha^{\frac{3}{2}}}^{\alpha}x\left\{e^{x^2}-2\alpha^2e^{\alpha^2}\log x-e^{\alpha^2}\left(1-2\alpha^2\log\alpha\right)\right\}dx \\ &=\pi\left[e^{x^2}\right]_{\alpha^{\frac{3}{2}}}^{\alpha}-\pi\alpha^2e^{\alpha^2}\left[2x^2\log x-x^2\right]_{\alpha^{\frac{3}{2}}}^{\alpha} \\ &\qquad -\pi e^{\alpha^2}\left(1-2\alpha^2\log\alpha\right)\left(\alpha^2-\alpha^3\right) \\ &=\pi e^{\alpha^2}\left(1-\alpha^2+\alpha^3+\alpha^4-\alpha^5+\alpha^5\log\alpha-e^{\alpha^3-\alpha^2}\right) \quad \text{(答)} \end{aligned}$$

問３　$F(t)=e^{-t}-1+t-\dfrac{t^2}{2}\ (0\leqq t\leqq 1)$

$G(t)=e^{-t}-1+t-\dfrac{t^2}{2}+\dfrac{t^3}{6}\ (0\leqq t\leqq 1)$ とおく．

$$F'(t)=-e^{-t}+1-t,\ F''(t)=e^{-t}-1\leqq 0$$

$y=F'(t)$ は $0\leqq t\leqq 1$ で単調減少．$F'(t)\leqq F'(0)=0$

$y=F(t)$ は $0\leqq t\leqq 1$ で単調減少．$F(t)\leqq F(0)=0$　　$\therefore\ F(t)\leqq 0$ ……①

$$G'(t)=-e^{-t}+1-t+\frac{t^2}{2},\ G''(t)=e^{-t}-1+t,\ G'''(t)=1-e^{-t}\geqq 0$$

$y=G''(t)$ は $0\leqq t\leqq 1$ で単調増加．$G''(t)\geqq G''(0)=0$

$y=G'(t)$ は $0\leqq t\leqq 1$ で単調増加．$G'(t)\geqq G'(0)=0$

$y=G(t)$ は $0\leqq t\leqq 1$ で単調増加．$G(t)\geqq G(0)=0$　　$\therefore\ G(t)\geqq 0$ ……②

以上①，②より，

$$-\frac{t^3}{6}\leqq e^{-t}-1+t-\frac{t^2}{2}\leqq 0\ (0\leqq t\leqq 1)$$ ……③ は示された．　（証明終）

問４　③ $\Leftrightarrow$ $1-t+\dfrac{t^2}{2}-\dfrac{t^3}{6}\leqq e^{-t}\leqq 1-t+\dfrac{t^2}{2}$ ……③′

③′ で $t=\alpha^2-\alpha^3$ $(0<\alpha<1$ より，$0<t<1)$ を代入．

$$1-\left(\alpha^2-\alpha^3\right)+\frac{1}{2}\left(\alpha^2-\alpha^3\right)^2-\frac{1}{6}\left(\alpha^2-\alpha^3\right)^3\leqq e^{\alpha^3-\alpha^2}\leqq 1-\left(\alpha^2-\alpha^3\right)+\frac{1}{2}\left(\alpha^2-\alpha^3\right)^2$$

これより，

$$\pi e^{\alpha^2}\left(\frac{1}{2}\alpha^4-\frac{1}{2}\alpha^6+\alpha^5\log\alpha\right)\leqq V(\alpha)\leqq \pi e^{\alpha^2}\left(\frac{1}{2}\alpha^4-\frac{1}{3}\alpha^6-\frac{1}{2}\alpha^7+\frac{1}{2}\alpha^8-\frac{1}{6}\alpha^9\right)$$

$$\Leftrightarrow\quad \pi e^{\alpha^2}\frac{\frac{1}{2}-\frac{1}{2}\alpha^2+\alpha\log\alpha}{\alpha^{c-4}}\leqq\frac{V(\alpha)}{\alpha^c}\leqq \pi e^{\alpha^2}\cdot\frac{\frac{1}{2}-\frac{1}{3}\alpha^2-\frac{1}{2}\alpha^3+\frac{1}{2}\alpha^4-\frac{1}{6}\alpha^5}{\alpha^{c-4}}$$

$\displaystyle\lim_{\alpha\to+0}\frac{V(\alpha)}{\alpha^c}$ が正の値に有限確定するためには，$c=4$ が必要であり，このとき，

はさみうち原理より，

$$\lim_{\alpha\to+0}\frac{V(\alpha)}{\alpha^c}=\frac{\pi}{2} \quad \text{(答)} \qquad c=4 \quad \text{(答)}$$

$$\left(\begin{array}{ll} (0<c<4) & \displaystyle\lim_{\alpha\to+0}\frac{V(\alpha)}{\alpha^c}=0 \\ (c>4) & \displaystyle\lim_{\alpha\to+0}\frac{V(\alpha)}{\alpha^c}=+\infty \end{array}\right)$$