I

イ. $\frac{5}{2}$　　ロ. $\frac{9}{2}$　　ハ. $\frac{29}{6}$

ニ. $90^\circ \leqq \theta \leqq 150^\circ$　　ホ. $\frac{37}{90}$

ヘ. $\frac{17}{8}$

II

(i) $\alpha = c + di$ (c, d：実数)とすると，

$(d \neq 0)$

$\alpha^2 - a\alpha + b = 0$

$\therefore\ (c+di)^2 - a(c+di) + b = 0$

$\therefore\ c^2 - d^2 - ac + b + i(2cd - ad) = 0$

$\therefore\ c^2 - d^2 - ac + b = 0,\ 2cd - ad = 0$ …㋐

ところで $x = c - di$ とすると

$x^2 - ax + b = c^2 - d^2 - ac + b$
$\quad - i(2cd - ad) = 0$ (∵㋐より)

よって $c - di$ も解 $c + di \neq c - di$

よって $\alpha = c + di$ なら $\beta = c - di$

よって題意は示せた。

(ii) $\beta - \alpha = b + ai - \beta$

$\therefore\ \alpha + b + ai = 2\beta\quad (=2\alpha)$

$\alpha = c + di$ とすると $\beta = c - di$

$c + di + b + ai = 2c - 2di$

$\therefore\ b + c = 2c,\ a + d = -2d$

$\therefore\ b = c,\ a = -3d$

また，解と係数の関係から

$\alpha + \beta = 2c = a,\quad \alpha\beta = c^2 + d^2 = b$

$\therefore\ d = -\frac{a}{3} = -\frac{2}{3}c$

$\therefore\ c^2 + \frac{4}{9}c^2 = c$

$c(13c - 9) = 0$

$c = 0,\ \frac{9}{13}$

$c = 0$ のとき $b = 0$, $a = 0$, $d = 0$ で
不適　(∵ $d \neq 0$)

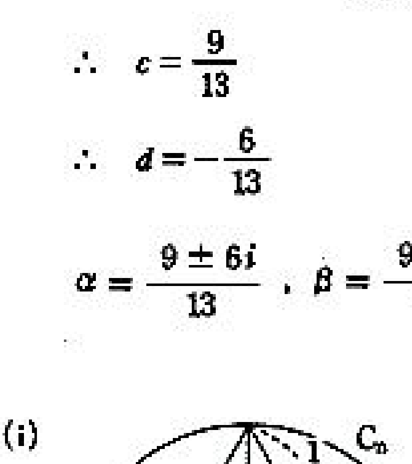
$\therefore\ c = \frac{9}{13}$

$\therefore\ d = -\frac{6}{13}$

$\alpha = \frac{9 \pm 6i}{13}$，$\beta = \frac{9 \mp 6i}{13}$ (複号同順) … (答)

III

(i)

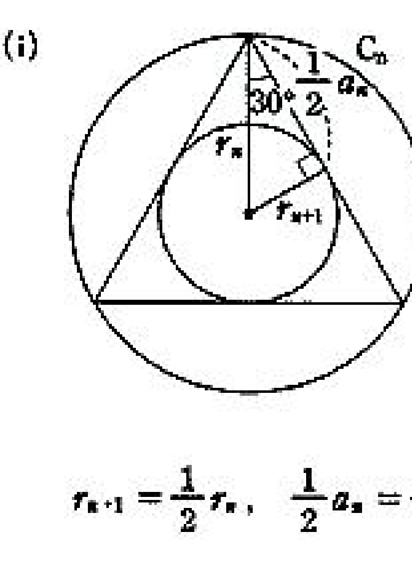


$r_{n+1} = \frac{1}{2}r_n$，$\frac{1}{2}a_n = \frac{\sqrt{3}}{2}r_n$

$r_{n+1} = \frac{1}{2}r_n = \cdots = \left(\frac{1}{2}\right)^n r_1 = \left(\frac{1}{2}\right)^n$

$\therefore\ r_n = \left(\frac{1}{2}\right)^{n-1}\quad \therefore\ a_n = \sqrt{3}\left(\frac{1}{2}\right)^{n-1}$ … (答)

(ii)



$A_n = C_n$ の面積 $- T_n$ の面積

$= \pi r^2{}_n - \frac{\sqrt{3}}{4}a_n^2$

$= \pi\left(\frac{1}{2}\right)^{2n-2} - \frac{\sqrt{3}}{4}\cdot 3\left(\frac{1}{2}\right)^{2n-2}$

$= \left(\pi - \frac{3\sqrt{3}}{4}\right)\left(\frac{1}{2}\right)^{2n-2}$

$= (4\pi - 3\sqrt{3})\left(\frac{1}{4}\right)^n$

$S_n = (4\pi - 3\sqrt{3})\sum_{k=1}^{n}\left(\frac{1}{4}\right)^k$

$= (4\pi - 3\sqrt{3})\frac{\frac{1}{4}\left\{1 - \left(\frac{1}{4}\right)^n\right\}}{1 - \frac{1}{4}}$

$= \frac{4\pi - 3\sqrt{3}}{3}\left\{1 - \left(\frac{1}{4}\right)^n\right\}$ … (答)

(iii) $A_1 = \pi - \frac{3\sqrt{3}}{4}$

よって

$\frac{3S_n}{4A_1} = \frac{4\pi - 3\sqrt{3}}{4\pi - 3\sqrt{3}}\left\{1 - \left(\frac{1}{4}\right)^n\right\} = 1 - \left(\frac{1}{4}\right)^n$

よって $\left(\frac{1}{4}\right)^n$ が小数第11位に

はじめて0でない数が出てくれば良い。

$\therefore\ -11 < n\log_{10}\left(\frac{1}{4}\right) < -10$

$\therefore\ -11 < -n\log_{10}4 < -10$

$\therefore\ 11 > n2\log_{10}2 > 10$

$\frac{11}{2}\log_2 10 > n > 5\log_2 10$

$\frac{11}{2} \times \log_2 10 = 18.271$

$5\log_2 10 = 16.61$

より $n = 17,\ 18$,

最小の n なので 17 … (答)

IV

(i) P での $y = x^2$ の接線の傾き $2a$

Q での $y = 4x - x^2$ の接線の傾き $4 - 2b$

$\therefore\ 2a = 4 - 2b\quad \therefore\ a = 2 - b$ … (答)

(ii) $\overrightarrow{PQ} = (b - a,\ 4b - b^2 - a^2)$

$= (2b - 2,\ 4b - b^2 - (2-b)^2)$

$= (2(b-1),\ -2b^2 + 8b - 4)$

直線PQ：

$(x - a,\ y - a^2) \mathbin{/\!/} (2b - 2,\ -2b^2 + 8b - 4)$

$\therefore\ x - a = k(2b - 2)$

$\quad y - a^2 = k(-2b^2 + 8b - 4)$

$\therefore\ (b^2 - 4b + 2)(x - a)$
$\qquad + (b - 1)(y - a^2) = 0$

$(b^2 - 4b + 2)(x - 2 + b)$
$\qquad + (b - 1)\{y - (b-2)^2\} = 0$

$(b^2 - 4b + 2)x + (b-2)(b^2 - 4b + 2)$
$+ (b-1)y - (b-2)(b^2 - 3b + 2) = 0$

$\therefore\ (b^2 - 4b + 2)x + (b-1)y$
$+ (b-2)(b^2 - 4b + 2 - b^2 + 3b - 2) = 0$

$\therefore\ (b^2 - 4b + 2)x + (b-1)y$
$\qquad - (b^2 - 2b) = 0$

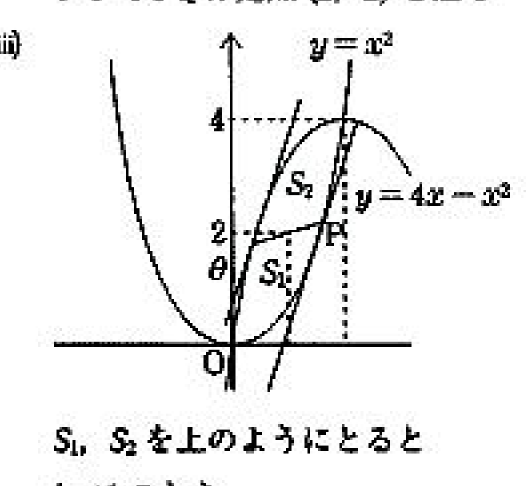
$\therefore\ (b^2 - 4b + 2)(x - 1)$
$\qquad + (b - 1)(y - 2) = 0$

よって $x = 1$, $y = 2$ は，

b にかかわらずこの式をみたす

よってPQは定点 (1, 2) を通る

(iii)



S_1, S_2 を上のようにとると

$b < 1$ のとき，

$S_1 = \int_0^b (4x - 2x^2)\,dx$
$\quad + \int_b^a \left\{\frac{(b^2 - 4b + 2)}{b - 1}x + \frac{b^2 - 2b}{b - 1} - x^2\right\}dx$

$= \left[2x^2 - \frac{2}{3}x^3\right]_0^b$
$\quad + \left[-\frac{1}{3}x^3 - \frac{b^2 - 4b + 2}{2(b-1)}x^2 + \frac{b^2 - 2b}{b - 1}x\right]_b^{2-b}$

$= 2b^2 - \frac{2}{3}b^3 - \frac{1}{3}(2-b)^3 - \frac{b^2 - 4b + 2}{2(b-1)}(2-b)^2$
$\quad + \frac{b^2 - 2b}{b - 1}(2 - b) + \frac{b^3}{3}$
$\quad + \frac{b^2 - 4b + 2}{2(b-1)}b^2 - \frac{b(b^2 - 2b)}{b - 1}$

$= 2b^2 - \frac{1}{3}\{b^3 + (2-b)^3\}$
$\quad - \frac{b^2 - 4b + 2}{2(b-1)}\{(2-b)^2 - b^2\}$
$\quad + \frac{b^2 - 2b}{b - 1}(2 - b - b)$

$= 2b^2 - \frac{1}{3}\cdot 2\{b^2 - b(2-b) + (2-b)^2\}$
$\quad + 2b^2 - 8b + 4 - 2b^2 + 4b$

$= 2b^2 - \frac{2}{3}(3b^2 - 6b + 4) - 4b + 4 = 2b^2 - 2b^2$
$\quad + 4b - \frac{8}{3} - 4b + 4 = \frac{4}{3}$

$S_2 = \int_b^{2-b}\left\{4x - x^2 + \frac{b^2 - 4b + 2}{b - 1}x - \frac{b^2 - 2b}{b - 1}\right\}dx$
$\quad + \int_{2-b}^2 (4x - 2x)\,dx$

$= \left[2x^2 - \frac{x^3}{3} + \frac{b^2 - 4b + 2}{2(b-1)}x^2 - \frac{b^2 - 2b}{b - 1}x\right]_b^{2-b}$
$\quad + \left[2x^2 - \frac{2}{3}x^3\right]_{(2-b)}^2$

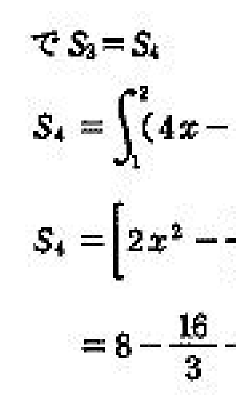
$= 2\{(2-b)^2 - b^2\} - \frac{1}{3}\{(2-b)^3 - b^3\}$
$\quad + \frac{b^2 - 4b + 2}{2(b-1)}\{(2-b)^2 - b^2\}$
$\quad - \frac{b^2 - 2b}{b - 1}(2 - b - b) + 8 - \frac{16}{3}$
$\quad - 2(2-b)^2 + \frac{2}{3}(2-b)^3$

$= -2b^2 + \frac{1}{3}\{b^2 - b(2-b) + (2-b)^2\} + 4b - \frac{4}{3}$

$= -2b^2 + \frac{1}{3}\cdot 2\cdot\{b^2 - b(2-b) + (2-b)^2\} + 4b - \frac{4}{3}$

$= -2b^2 + 2b^2 - 4b + \frac{8}{3} + 4b - \frac{4}{3} = \frac{4}{3}\qquad \therefore\ S_1 = S_2$

$b = 1$ のときにはPQ：$x = 1$



$S_3 = \int_0^1 (4x - 2x^2)\,dx$

$S_3 = \left[2x^2 - \frac{2}{3}x^3\right]_0^1$

$= 2 - \frac{2}{3} = \frac{4}{3}$

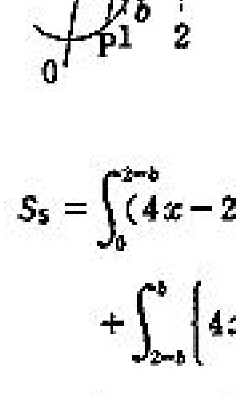
で $S_3 = S_4$

$S_4 = \int_1^2 (4x - 2x^2)\,dx$

$S_4 = \left[2x^2 - \frac{2}{3}x^3\right]_1^2$

$= 8 - \frac{16}{3} - 2 + \frac{2}{3} = \frac{4}{3}$

$b > 1$ のとき



$S_5 = \int_0^{2-b}(4x - 2x^2)\,dx$
$\quad + \int_{2-b}^b \left\{4x - x^2 + \frac{b^2 - 4b + 2}{b + 1}x - \frac{b^2 - 2b}{b - 1}\right\}dx$

$= \left[2x^2 - \frac{2}{3}x^3\right]_0^{2-b}$
$\quad + \left[2x^2 - \frac{x^3}{3} + \frac{b^2 - 4b + 2}{2(b-1)}x^2 - \frac{b^2 - 2b}{b - 1}x\right]_{2-b}^b$

$= 2(2-b)^2 - \frac{2}{3}(2-b)^3 + 2b^2$
$\quad - \frac{b^3}{3} + \frac{b^2 - 4b + 2}{2(b-1)}\{b^2 - (2-b)^2\}$
$\quad - \frac{b^2 - 2b}{b - 1}(b - 2 + b) - 2(2-b)^2 + \frac{1}{3}(2-b)^3$

$= -\frac{1}{3}\{(2-b)^3 + b^3\} + 2b^2 + 2b^2 - 8b + 4 - 2b^2 + 4b$

$= -\frac{1}{3}(2 - b + b)\{(2-b)^2 - b(2-b) + b^2\} + 2b^2 - 4b + 4$

$= -\frac{2}{3}(3b^2 - 6b + 4) + 2b^2 - 4b + 4$

$= -\frac{8}{3} + 4 = \frac{4}{3}$

$S_6 = \int_{2-b}^b \left\{-\frac{b^2 - 4b + 2}{b - 1}x + \frac{b^2 - 2b}{b - 1} - x^2\right\}dx$
$\quad + \int_b^2 (4x - 2x^2)\,dx$

$= \left[-\frac{b^2 - 4b + 2}{2(b-1)}x^2 + \frac{b^2 - 2b}{b - 1}x - \frac{1}{3}x^3\right]_{2-b}^b$
$\quad + \left[2x^2 - \frac{2}{3}x^3\right]_b^2$

$= -\frac{b^2 - 4b + 2}{2(b-1)}\{b^2 - (2-b)^2\}$
$\quad + \frac{b^2 - 2b}{b - 1}(b - 2 + b) - \frac{1}{3}b^3 + \frac{1}{3}b(2-b)^3$
$\quad + 8 - \frac{16}{3} - 2b^2 + \frac{2}{3}b^3$

$= -2b^2 + 8b - 4 + 2b^2 - 4b + \frac{1}{3}\{b^3 + (2-b)^3\} + \frac{8}{3} - 2b^2$

$= 4b - \frac{4}{3} - 2b^2 + \frac{1}{3}(b + 2 - b)\{b^2 - b(2-b) + (2-b)^2\}$

$= 4b - \frac{4}{3} - 2b^2 + \frac{2}{3}(3b^2 - 6b + 4)$

$= \frac{4}{3}$

やはり $S_5 = S_6$

以上よりPQは2つの放物線に囲まれる部分の面積を二等分する。