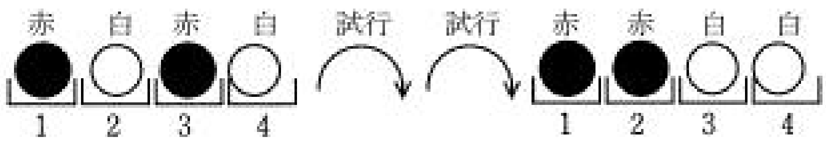
1

(1)

$$\begin{cases} p\,(\text{同じ色の玉を入れ替える}) = \dfrac{2}{{}_4C_2} = \dfrac{1}{3} & \cdots\cdots\text{(A)} \\ p\,(\text{異なる色の玉を入れ替える}) = 1 - \dfrac{1}{3} = \dfrac{2}{3} & \cdots\cdots\text{(B)} \end{cases}$$

$$p = \underbrace{\left(\frac{1}{3}\right)^2}_{\text{(A)}\to\text{(A)}} + \underbrace{\frac{2}{3}\cdot\frac{1}{{}_4C_2}}_{\text{(B)}\to\text{((B)を打ち消す)}} = \frac{1}{9} + \frac{1}{9} = \boxed{\frac{2}{9}}^{(ア)}\ \text{(答)}$$



$$p\,(2 \leftrightarrow 3\ \text{を入れ替える}) = \frac{1}{{}_4C_2} = \frac{1}{6} \qquad \cdots\cdots\text{(C)}$$

$$p = \underbrace{\frac{1}{6}\cdot\frac{1}{3}}_{\text{(C)}\to\text{(A)}} + \underbrace{\frac{1}{3}\cdot\frac{1}{6}}_{\text{(A)}\to\text{(C)}} + \underbrace{\frac{1}{6}}_{(3\leftrightarrow 4)}\cdot\underbrace{\frac{1}{6}}_{(2\leftrightarrow 4)} + \underbrace{\frac{1}{6}}_{(1\leftrightarrow 2)}\cdot\underbrace{\frac{1}{6}}_{(1\leftrightarrow 3)} = \boxed{\frac{1}{6}}^{(イ)}\ \text{(答)}$$

(2)

$$\begin{aligned} z &= \cos\theta + i\sin\theta + \sqrt{3}(i\cos\theta - \sin\theta) \\ &= \cos\theta - \sqrt{3}\sin\theta + i(\sin\theta + \sqrt{3}\cos\theta) \end{aligned}$$

ここで，

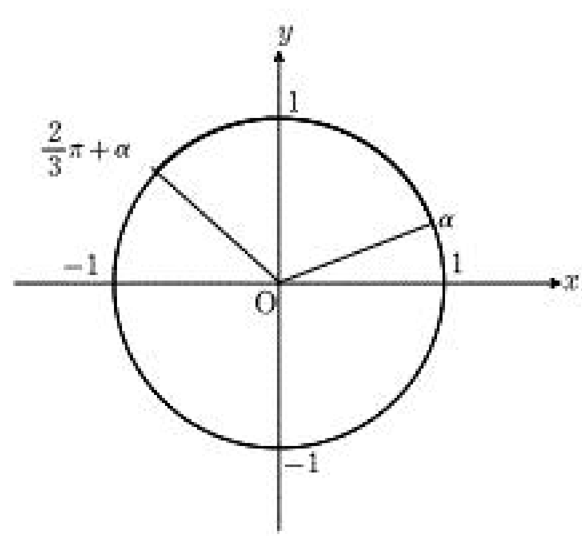
$$\sqrt{2}z - 1 + i = \sqrt{2}\cos\theta - \sqrt{6}\sin\theta - 1 + i(\sqrt{2}\sin\theta + \sqrt{6}\cos\theta + 1)$$

よって，

$$\begin{aligned} \left|\sqrt{2}z - 1 + i\right|^2 &= (\sqrt{2}\cos\theta - \sqrt{6}\sin\theta - 1)^2 + (\sqrt{2}\sin\theta + \sqrt{6}\cos\theta + 1)^2 \\ &= 2\sqrt{2}\left\{(\sqrt{3}+1)\sin\theta + (\sqrt{3}-1)\cos\theta\right\} + 10 \\ &= 8\sin(\theta+\alpha) + 10 \\ &\left(\text{ただし，}\cos\alpha = \frac{\sqrt{3}+1}{2\sqrt{2}},\ \sin\alpha = \frac{\sqrt{3}-1}{2\sqrt{2}}\right) \end{aligned}$$

$0 \leqq \theta \leqq \dfrac{2}{3}\pi$ より，

$$\alpha \leqq \theta + \alpha \leqq \frac{2}{3}\pi + \alpha$$



よって，

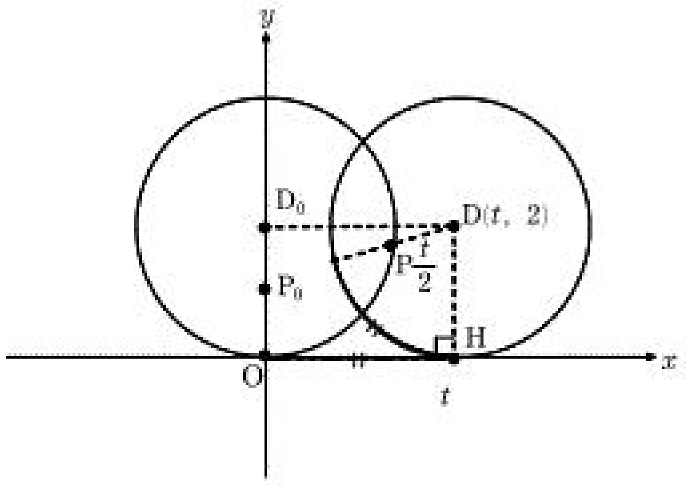
$$\left(\left|\sqrt{2}z - 1 + i\right|^2 \text{の最大値}\right) = \underbrace{8}_{\{\sin(\theta+\alpha)=1\text{のとき}\}} + 10 = 18$$

$$\therefore\ \left(\left|\sqrt{2}z - 1 + i\right| \text{の最大値}\right) = \boxed{3\sqrt{2}}^{(ウ)}\ \text{(答)}$$

2

(1)

(i)



$$\overrightarrow{OP}=(x,\ y)$$
$$=(t,\ 2)+\left(\cos\left(\frac{3}{2}\pi-\frac{t}{2}\right),\ \sin\left(\frac{3}{2}\pi-\frac{t}{2}\right)\right)$$
$$=(t,\ 2)+\left(-\sin\frac{t}{2},\ -\cos\frac{t}{2}\right)$$
$$\therefore\ (x,\ y)=\left(\boxed{t-\sin\frac{t}{2}}_{(エ)},\ \boxed{2-\cos\frac{t}{2}}_{(オ)}\right)\ \text{(答)}$$

(ii)

$$\begin{cases}\dfrac{dx}{dt}=1-\dfrac{1}{2}\cos\dfrac{t}{2}\\[2mm] \quad =\dfrac{2-\cos\frac{t}{2}}{2}\ (>0)\\[2mm] \dfrac{dy}{dt}=\dfrac{1}{2}\sin\dfrac{t}{2}\end{cases}$$

$\vec{v}=\left(\dfrac{dx}{dt},\ \dfrac{dy}{dt}\right)$ とする．

$t\geqq 0$ での増減表は右に表記．

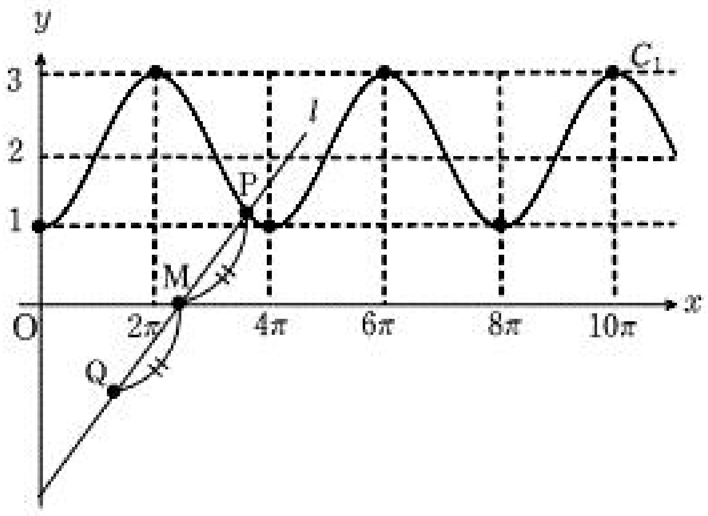
t	0	$\cdots$	2π	$\cdots$	4π	$\cdots$	6π	$\cdots\cdots$
$\frac{dx}{dt}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+\cdots$
x	0		2π		4π		6π	$\cdots$
$\frac{dy}{dt}$	0	$+$	0	$-$	0	$+$	0	$-\cdots$
y	1		3		1		3	$\cdots$
$(x,\ y)$	$(0,\ 1)$		$(2\pi,\ 3)$		$(4\pi,\ 1)$		$(6\pi,\ 3)$	$\cdots$
$\vec{v}$	→	↗	→	↘	→	↗	→	↘ $\cdots$

(答)

(2)

$$l:y=-\frac{2-\cos\frac{t}{2}}{\sin\frac{t}{2}}\left(x-t+\sin\frac{t}{2}\right)+2-\cos\frac{t}{2}$$
$$\Longleftrightarrow\ y=-\frac{2-\cos\frac{t}{2}}{\sin\frac{t}{2}}x+\frac{t\left(2-\cos\frac{t}{2}\right)}{\sin\frac{t}{2}}$$

（ただし，$t\neq 2n\pi$ （n:自然数)）



ちなみに，

$$\left.\begin{array}{l}t=0\ \cdots\cdots\ Q\left(0,\ -1\right)\\ t=2\pi\ \cdots\cdots\ Q\left(2\pi,\ -3\right)\\ t=4\pi\ \cdots\cdots\ Q\left(4\pi,\ -1\right)\\ t=6\pi\ \cdots\cdots\ Q\left(6\pi,\ -3\right)\\ \quad\vdots\qquad\qquad\quad\vdots\end{array}\right|\cdots\cdots①$$

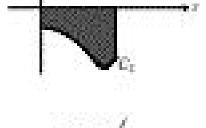
l において，$y=0$ とおくと，　$x=t$ 　より，$M\left(t,\ 0\right)$

$$P\left(t-\sin\frac{t}{2},\ 2-\cos\frac{t}{2}\right),\ \underline{\underline{Q\left(x,\ y\right)}},\ M\left(t,\ 0\right)$$

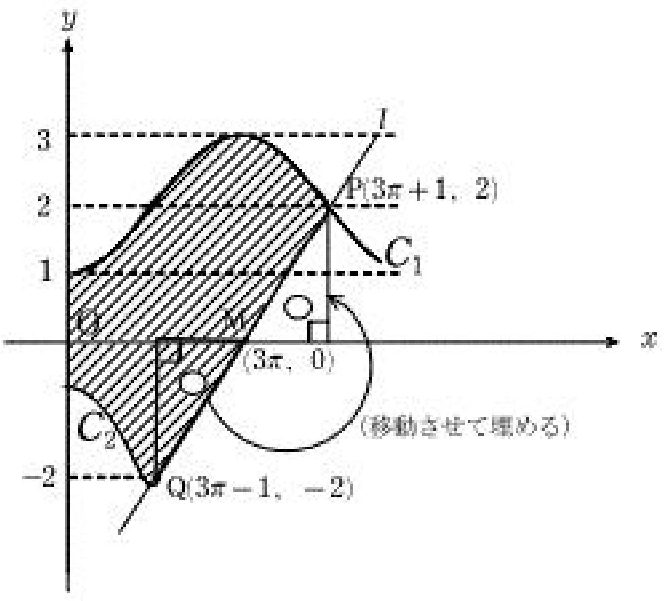
点 M は 2 点 P，Q の中点より，

$$\begin{cases}\dfrac{t-\sin\frac{t}{2}+x}{2}=t\\[2mm] \dfrac{2-\cos\frac{t}{2}+y}{2}=0\end{cases}\qquad \therefore\ \underline{Q\left(t+\sin\frac{t}{2},\ \cos\frac{t}{2}-2\right)\ \text{(答)}}\qquad \text{(①をすべて満たす)}$$

(3) Q の x 座標について，　$\dfrac{dx}{dt}=1+\dfrac{1}{2}\cos\dfrac{t}{2}\ (>0)$

$S=$  $+$ 

$$=\int_0^{3\pi+1}\left(2-\cos\frac{t}{2}\right)dt+\int_0^{3\pi-1}\left(2-\cos\frac{t}{2}\right)dt$$
$$=\int_0^{3\pi}\left(2-\cos\frac{t}{2}\right)\frac{2-\cos\frac{t}{2}}{2}dt$$
$$+\int_0^{3\pi}\left(2-\cos\frac{t}{2}\right)\frac{2+\cos\frac{t}{2}}{2}dt$$
$$=\int_0^{3\pi}\left(4-2\cos\frac{t}{2}\right)dt$$
$$=\left[4t-4\sin\frac{t}{2}\right]_0^{3\pi}$$
$$=\underline{12\pi+4\ \text{(答)}}$$



3

(1)　$a=2l+1$（l:自然数）と表せる．

$ax-2y=1 \iff (2l+1)x-2y=1$　……①

ここで，

$$\begin{array}{rl} & (2l+1)x-2y=1 \\ -) & (2l+1)\cdot 1-2\cdot l=1 \\ \hline & (2l+1)(x-1)-2(y-l)=0 \end{array}$$

ゆえに，

$(2l+1)(x-1)=2(y-l)$　←（①と同じ）

$2l+1$ と 2 は互いに素な整数だから，

$$\begin{cases} x-1=2k \\ y-l=(2l+1)k \end{cases} \iff \underbrace{\begin{cases} x=2k+1 \\ y=(2l+1)k+l \end{cases}}_{②} \quad (k \text{は} 0 \text{以上の整数})$$

②より，0 以上の整数 k の数だけ (x, y) の組が存在するので，(x, y) の組は無数に存在する．

（証明終）

(2)　十分大きな n に対して，

$$\begin{cases} x_k=2(k-1)+1=2k-1 \\ y_k=(2l+1)(k-1)+l=(2l+1)k-l-1 \end{cases} \quad (k=1, 2, 3, \cdots\cdots)$$

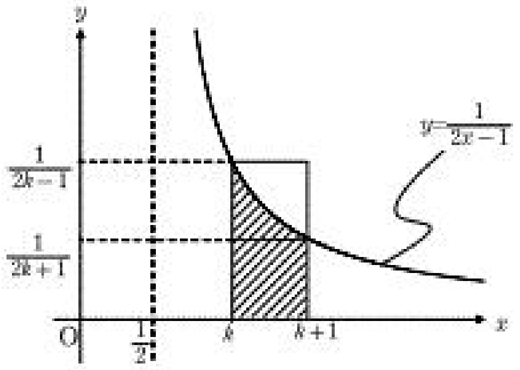
と考えてよい．

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n}\frac{y_k}{x_k} &= \frac{1}{n}\sum_{k=1}^{n}\frac{(2l+1)k-l-1}{2k-1} \\ &= \frac{1}{n}\sum_{k=1}^{n}\left(l+\frac{1}{2}+\frac{-\frac{1}{2}}{2k-1}\right) \\ &= l+\frac{1}{2}-\frac{1}{2n}\sum_{k=1}^{n}\frac{1}{2k-1} \end{aligned}$$

$$\therefore \lim_{n\to\infty}\sum_{k=1}^{n}\frac{y_k}{x_k}=l+\frac{1}{2}-\frac{1}{2}\underbrace{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2k-1}}_{③}$$

ここで，$y=\dfrac{1}{2x-1}\left(x>\dfrac{1}{2}\right)$ は単調減少．

面積を用いて，区間 $[k, k+1]$ において



$$\underbrace{\frac{1}{2k+1}<\underbrace{\int_k^{k+1}\frac{dx}{2x-1}}_{④}}_{⑤}<\frac{1}{2k-1}$$　と表せる．

④より，$k=1, 2, 3, \cdots\cdots, n$ を代入し，辺々和をとると，

$$\int_1^{n+1}\frac{dx}{2x-1}<\sum_{k=1}^{n}\frac{1}{2k-1} \iff \frac{1}{2}\log(2n+1)<\sum_{k=1}^{n}\frac{1}{2k-1} \quad \cdots\cdots ④'$$

⑤より，$k=1, 2, 3, \cdots\cdots, n-1$ まで代入し，辺々和をとると，

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\cdots+\frac{1}{2n-1}<\int_1^{n}\frac{dx}{2x-1}$$

$$\iff \sum_{k=1}^{n}\frac{1}{2k-1}<\frac{1}{2}\log(2n-1)+1 \quad \cdots\cdots ⑤'$$

④′, ⑤′ より，

$$\frac{1}{2}\log(2n+1)<\sum_{k=1}^{n}\frac{1}{2k-1}<\frac{1}{2}\log(2n-1)+1$$

$$\iff \underbrace{\frac{2n+1}{2n}}_{\to 1}\cdot\underbrace{\frac{\log(2n+1)}{2n+1}}_{\to 0}<\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2k-1}$$

$$<\underbrace{\frac{2n-1}{2n}}_{\to 1}\cdot\underbrace{\frac{\log(2n-1)}{2n-1}}_{\to 0}+\underbrace{\frac{1}{n}}_{\to 0}$$

(最左辺)
(最右辺) $\to 0\ (n\to\infty)$

はさみうちの原理より，$\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2k-1}=0$

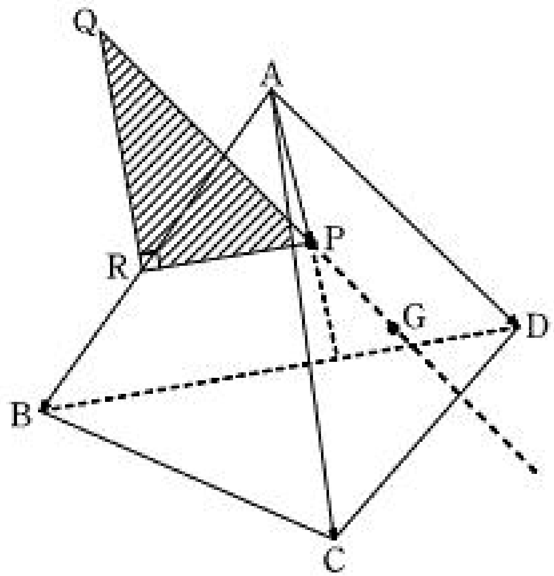
以上より，

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{y_k}{x_k} &= l+\frac{1}{2} \quad (③\text{より}) \\ &= \underline{\frac{a}{2}} \quad (\text{答}) \end{aligned}$$

4

正四面体の各辺の長さを全て1としてよい．

$$\begin{cases} \left|\overrightarrow{AB}\right| = \left|\overrightarrow{AC}\right| = \left|\overrightarrow{AD}\right| = 1 \\ \overrightarrow{AB}\cdot\overrightarrow{AC} = \overrightarrow{AC}\cdot\overrightarrow{AD} = \overrightarrow{AD}\cdot\overrightarrow{AB} = \dfrac{1}{2} \end{cases} \quad \cdots\cdots ①$$



$\overrightarrow{AR} = k\overrightarrow{AB}$ $(0 \leqq k \leqq 1)$ と表せる．

$$\overrightarrow{AQ} = (1-t)\overrightarrow{AP} + t\overrightarrow{AG}$$

$$= (1-t)\left(\frac{1}{8}\overrightarrow{AB} + \frac{1}{4}\overrightarrow{AD}\right) + t\frac{\overrightarrow{AC}+\overrightarrow{AD}}{3}$$

$$= \frac{1-t}{8}\overrightarrow{AB} + \frac{t}{3}\overrightarrow{AC} + \frac{t+3}{12}\overrightarrow{AD} \quad (t:\text{実数})$$

となり，$\dfrac{t+3}{12} = 0$　ゆえに，$t = -3$

よって，$\overrightarrow{AQ} = \dfrac{1}{2}\overrightarrow{AB} - \overrightarrow{AC}$

ここで，

$$\begin{cases} \overrightarrow{RQ} = \overrightarrow{AQ} - \overrightarrow{AR} = \left(\dfrac{1}{2} - k\right)\overrightarrow{AB} - \overrightarrow{AC} \\ \overrightarrow{RP} = \overrightarrow{AP} - \overrightarrow{AR} = \left(\dfrac{1}{8} - k\right)\overrightarrow{AB} + \dfrac{1}{4}\overrightarrow{AD} \end{cases}$$

$$\overrightarrow{RQ}\cdot\overrightarrow{RP} = \left\{\left(\frac{1}{2}-k\right)\overrightarrow{AB} - \overrightarrow{AC}\right\}\cdot\left\{\left(\frac{1}{8}-k\right)\overrightarrow{AB} + \frac{1}{4}\overrightarrow{AD}\right\} \quad ①\text{より}$$

$$= \left(\frac{1}{2}-k\right)\left(\frac{1}{8}-k\right) + \frac{1}{4}\left(\frac{1}{2}-k\right)\cdot\frac{1}{2} - \left(\frac{1}{8}-k\right)\cdot\frac{1}{2} - \frac{1}{4}\cdot\frac{1}{2} = 0$$

$$\iff k^2 - \frac{1}{4}k - \frac{1}{16} = 0$$

$$\iff 16k^2 - 4k - 1 = 0$$

$$\iff k = \frac{1 \pm \sqrt{5}}{8}$$

ここで，$0 \leqq k \leqq 1$ より，$k = \dfrac{1+\sqrt{5}}{8}$ $\left(= \dfrac{AR}{AB}\right)$（答）