1

(1)

$$\left|\overrightarrow{AB}\right|^2=\left|\overrightarrow{OB}-\overrightarrow{OA}\right|^2=\left|\overrightarrow{OB}\right|^2-2\overrightarrow{OA}\cdot\overrightarrow{OB}+\left|\overrightarrow{OA}\right|^2 \quad \cdots\cdots①$$

$$3\overrightarrow{OA}+7\overrightarrow{OB}+5\overrightarrow{OC}=\vec{0} \iff 3\overrightarrow{OA}+7\overrightarrow{OB}=-5\overrightarrow{OC}$$

$$\left|3\overrightarrow{OA}+7\overrightarrow{OB}\right|=5\left|\overrightarrow{OC}\right|$$

両辺2乗して

$$9\left|\overrightarrow{OA}\right|^2+42\overrightarrow{OA}\cdot\overrightarrow{OB}+49\left|\overrightarrow{OB}\right|^2=25\left|\overrightarrow{OC}\right|^2 \quad \cdots\cdots②$$

$\left|\overrightarrow{OA}\right|=\left|\overrightarrow{OB}\right|=\left|\overrightarrow{OC}\right|=1$ より

$$58+42\overrightarrow{OA}\cdot\overrightarrow{OB}=25 \qquad \therefore\ \overrightarrow{OA}\cdot\overrightarrow{OB}=-\frac{11}{14}$$

①に代入し

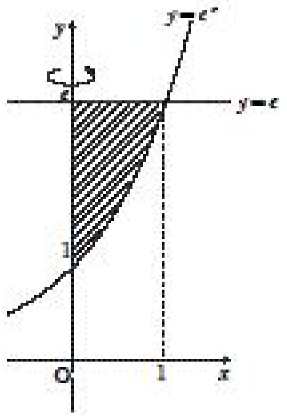
$$\left|\overrightarrow{AB}\right|^2=2+\frac{11}{7}=\frac{25}{7} \qquad \therefore\ \left|\overrightarrow{AB}\right|=\frac{5}{\sqrt{7}} \quad \cdots\cdots(答)$$

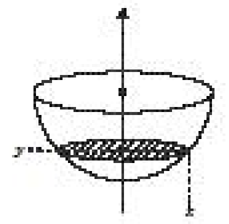
(2)

$y=e^x$

$\iff x=\log y$

$\dfrac{dy}{dx}=e^x$





y	$1 \to e$
x	$0 \to 1$

求める体積 V は,

$$V=\int_1^e \pi x^2\,dy=\int_1^e \pi(\log y)^2\,dy$$

$$=\int_0^1 \pi x^2\cdot e^x\,dx \quad \cdots\cdots①$$

$$\int_0^1 x^2e^x\,dx=\left[x^2e^x\right]_0^1-\int_0^1 2x\cdot e^x\,dx=\left[x^2e^x\right]_0^1-\left(\left[2x\cdot e^x\right]_0^1-\int_0^1 2\cdot e^x d\right)$$

$$=e-2e+2e-2=e-2$$

①より

$V=\underline{\pi(e-2)}$ ……(答)

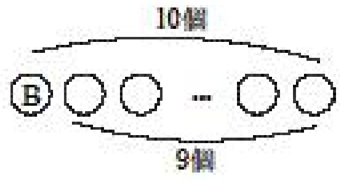
(3)

B, W 合わせて10個であり, W同士は隣り合わないので,

Wの数は0～5個.

先頭はBなので, 残りの9個の内訳は,

(B, W)＝(9, 0), (8, 1)…(4, 5)



(i) (B, W)＝(9, 0) … 1通り

(ii) (B, W)＝(8, 1) … 9通り

(iii) (B, W)＝(7, 2) … 先に7個Bを並べてWに入れる $_8C_2=28$ 通り

(iv) (B, W)＝(6, 3) … (iii)と同様に $_7C_3=35$ 通り

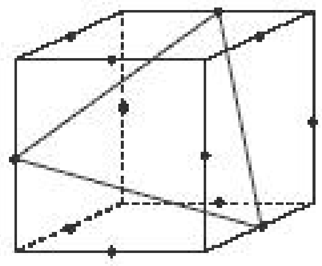
(v) (B, W)＝(5, 4) … (iii)と同様に $_6C_4=15$ 通り

(vi) (B, W)＝(4, 5) … (iii)と同様に $_5C_5=1$ 通り

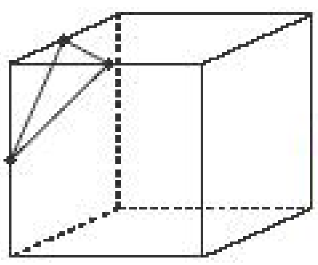
(i)～(vi)より, $\underline{89通り}$ ……(答)

(4)

(i) 正三角形

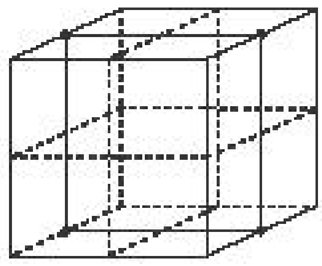


… 8個 $\left(1辺\dfrac{\sqrt{6}}{2}\right)$

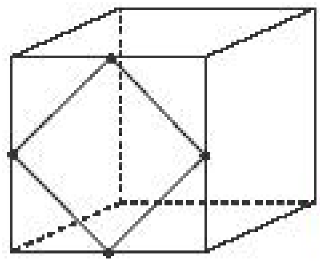


… 8個 $\left(1辺\dfrac{\sqrt{2}}{2}\right)$

(ii) 正方形

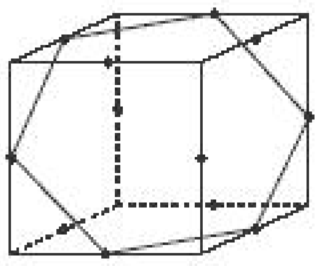


… 3個 (1辺1)



… 6個 $\left(1辺\dfrac{\sqrt{2}}{2}\right)$

(iii) 正六角形



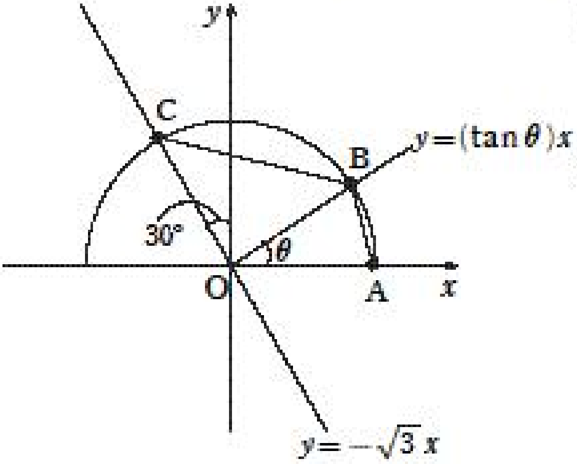
… 4個

よって,

$8+8+3+6+4=\underline{29個}$ ……(答)

2

(1)　半径1の円に内接する正6角形の面積の $\frac{1}{3}$ は $\frac{\sqrt{3}}{2}$ であるから，



$$S_{OABC}=S_{OAB}+S_{OBC}$$

$$=\frac{1}{2}r^2\sin\theta+\frac{1}{2}r^2\sin\left(\frac{2}{3}\pi-\theta\right)$$

$$\frac{\sqrt{3}}{2}=\frac{1}{2}r^2\left(\sin\theta+\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta\right)$$

$$=\frac{1}{2}r^2\left(\frac{3}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta\right)$$

$$=\frac{1}{4}r^2\left(3\sin\theta+\sqrt{3}\cos\theta\right)$$

$$=\frac{\sqrt{3}}{2}r^2\sin\left(\theta+\frac{\pi}{6}\right)$$

$$\therefore r^2=\frac{1}{\sin\left(\theta+\frac{\pi}{6}\right)}$$　……(答)

(2)

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r^2\,d\theta=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\sin\left(\theta+\frac{\pi}{6}\right)}d\theta=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin\left(\theta+\frac{\pi}{6}\right)}{\sin^2\left(\theta+\frac{\pi}{6}\right)}d\theta$$

ここで，$t=\cos\left(\theta+\frac{\pi}{6}\right)$ とする．

$$\left(\begin{array}{l}\theta:\frac{\pi}{6}\to\frac{\pi}{3}\text{ のとき，}t:\frac{1}{2}\to 0\\ \frac{dt}{d\theta}=-\sin\left(\theta+\frac{\pi}{6}\right)\end{array}\right)$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r^2\,d\theta=\int_{\frac{1}{2}}^{0}\frac{\sqrt{1-t^2}}{1-t^2}\cdot\left(-\frac{1}{\sqrt{1-t^2}}\right)dt$$

$$=\int_{\frac{1}{2}}^{0}\frac{-1}{1-t^2}dt$$

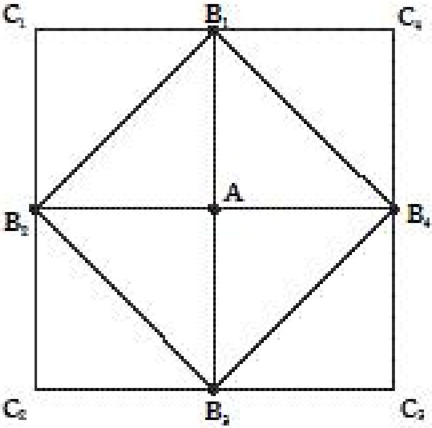
$$=\int_{\frac{1}{2}}^{0}\frac{1}{t^2-1}dt$$

$$=\int_{\frac{1}{2}}^{0}\frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=\frac{1}{2}\left\{\Big[\log|t-1|\Big]_{\frac{1}{2}}^{0}-\Big[\log|t+1|\Big]_{\frac{1}{2}}^{0}\right\}$$

$$=\frac{1}{2}\log 3$$　……(答)

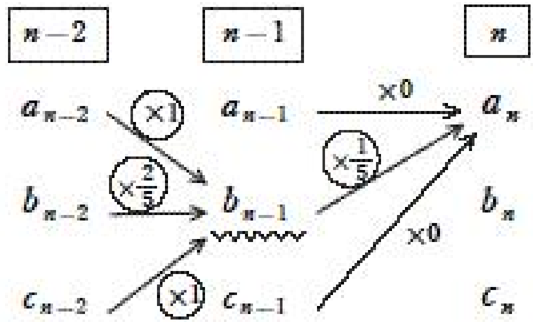
3



A にいる確率 … a_n

$B_{1\sim4}$ にいる確率 … b_n

$C_{1\sim4}$ にいる確率 … c_n



$$a_n=\frac{1}{5}b_{n-1}=\frac{1}{5}\left(\frac{2}{5}b_{n-2}+(a_{n-2}+b_{n-2})\right)$$

$$=\frac{1}{5}\left(\frac{2}{5}b_{n-2}+(1-c_{n-2})\right)$$

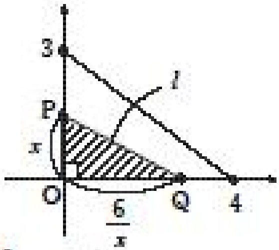
$$\therefore\ b_{n-1}=-\frac{3}{5}b_{n-2}+1 \iff b_n=\frac{3}{8}\cdot\left(-\frac{3}{5}\right)^{n-1}+\frac{5}{8}$$

よって，

$$\underline{a_n=-\frac{1}{8}\cdot\left(-\frac{3}{5}\right)^{n-1}+\frac{1}{8}} \quad \cdots\cdots(答)$$

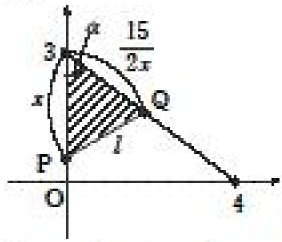
4

(i)



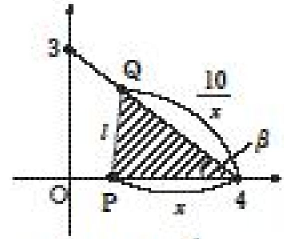
$\left(\frac{3}{2} \leqq x \leqq 3\right)$

(ii)



$\left(\frac{3}{2} \leqq x \leqq 3\right)$ $\cos\alpha = \frac{3}{5}$

(iii)



$(2 \leqq x \leqq 4)$ $\cos\beta = \frac{4}{5}$

(i) $l^2 = x^2 + \left(\frac{6}{x}\right)^2 = x^2 + \frac{36}{x^2} \geqq 2\sqrt{x^2 \cdot \frac{36}{x^2}} = 12$

$x = \frac{3}{2}$ のとき, $l^2 = \frac{73}{4}$

$x = 3$ のとき, $l^2 = 13$ $\qquad \therefore$ $l_{\min} = 2\sqrt{3}$, $l_{\max} = \frac{\sqrt{73}}{2}$

(ii) $l^2 = x^2 + \left(\frac{15}{2x}\right)^2 - 2 \cdot x \cdot \frac{15}{2x} \cdot \cos\alpha = x^2 + \frac{15^2}{4x^2} - 9 \geqq 2\sqrt{x^2 \cdot \frac{15^2}{4x^2}} - 9 = 6$

$x = \frac{3}{2}$ のとき, $l^2 = \frac{73}{4}$

$x = 3$ のとき, $l^2 = \frac{25}{4}$ $\qquad \therefore$ $l_{\min} = \sqrt{6}$, $l_{\max} = \frac{\sqrt{73}}{2}$

(iii) $l^2 = x^2 + \left(\frac{10}{x}\right)^2 - 2 \cdot x \cdot \frac{10}{x} \cdot \cos\beta = x^2 + \frac{10^2}{x^2} - 16 \geqq 2\sqrt{x^2 \cdot \frac{10^2}{x^2}} - 16 = 4$

$x = 2$ のとき, $l^2 = 13$

$x = 4$ のとき, $l^2 = \frac{25}{4}$ $\qquad \therefore$ $l_{\min} = \sqrt{2}$, $l_{\max} = \sqrt{13}$

以上, (i)～(iii)より, $l_{\min} = \sqrt{2}$, $l_{\max} = \frac{\sqrt{73}}{2}$ ……(答)