問 1

(1)

$$
\begin{array}{rl}
S = & 4\cdot4^0+7\cdot4^1+10\cdot4^2+\cdots\cdots+(3n+1)\cdot4^{n-1} \\
-)\ 4S = & \qquad\quad 4\cdot4^1+\ 7\cdot4^2+\cdots\cdots+(3n-2)\cdot4^{n-1}+(3n+1)\cdot4^n \\ \hline
-3S = & 4+3(4^1+4^2+4^3+\cdots\cdots+4^{n-1})-(3n+1)\cdot4^n \\
= & 4+3\cdot\dfrac{4(4^{n-1}-1)}{4-1}-(3n+1)\cdot4^n
\end{array}
$$

$\therefore\ \underline{S=n\cdot4^n\ (n=1,2,3,\cdots\cdots)}$ ……(答)

(2) $\left(\dfrac{1-i}{\sqrt{2}}\right)^2=-i$ より，

$$
\begin{aligned}
\sum_{k=1}^{2017}\left(\frac{1-i}{\sqrt{2}}\right)^{2k} &= \sum_{k=1}^{2017}(-i)^k \\
&= \underbrace{\{(-i)+(-1)+i+1\}}_{(4\text{つの周期的循環群})}\cdot504+(-i) \\
&= 0\cdot504-i \\
&= \underline{-i}
\end{aligned}
$$

……(答)

(3) $p_n=1-\left(\dfrac{5}{6}\right)^n$ より， $p_n\geqq0.95$

$$
\begin{aligned}
p_n\geqq0.95 &\Longleftrightarrow 1-\left(\frac{5}{6}\right)^n\geqq0.95 \\
&\Longleftrightarrow 0.05\geqq\left(\frac{5}{6}\right)^n \\
&\Longleftrightarrow \log_{10}\frac{5}{100}\geqq n\log_{10}\frac{5}{6} \\
&\Longleftrightarrow \log_{10}\frac{10}{2}-\log_{10}100\geqq n\left(\log_{10}\frac{10}{2}-\log_{10}2\cdot3\right) \\
&\Longleftrightarrow 1-0.3010-2\geqq n(1-0.3010-0.3010-0.4771) \\
&\Longleftrightarrow \underset{\substack{\shortparallel \\ 16.44\cdots\cdots}}{\frac{1.3010}{0.0791}}\leqq n
\end{aligned}
$$

$\therefore\ \underline{(\text{最小の}\ n)=17}$ ……(答)

(4)

$$
\begin{cases} A=x+y \\ B=x-y \end{cases}
\Longleftrightarrow
\begin{cases} x=\dfrac{A+B}{2} \\ y=\dfrac{A-B}{2} \end{cases}
$$

$$
\left.\begin{aligned}
\sin A=\frac{1}{\sqrt{3}}\quad\left(\cos^2A=\frac{2}{3}\right) \\
\cos B=\frac{2}{\sqrt{5}}\quad\left(\sin^2B=\frac{1}{5}\right)
\end{aligned}\right\}\ \cdots\cdots①
$$

ここで，

$$
\begin{aligned}
\sin2x\sin2y &= \sin(A+B)\sin(A-B) \\
&= (\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B) \\
&= (\sin A\cos B)^2-(\cos A\sin B)^2 \qquad (①\text{より}) \\
&= \frac{1}{3}\cdot\frac{4}{5}-\frac{2}{3}\cdot\frac{1}{5} \\
&= \frac{4}{15}-\frac{2}{15} \\
&= \underline{\frac{2}{15}}
\end{aligned}
$$

……(答)

問2

(1) $(X, Y, Z) = (1, 1, 2), (1, 1, 3), (1, 1, 4), (1, 1, 5), (1, 1, 6) \cdots\cdots (1, 1, n)$

$(2, 2, 3), (2, 2, 4), (2, 2, 5), (2, 2, 6) \cdots\cdots (2, 2, n)$

$(3, 3, 4), (3, 3, 5), (3, 3, 6) \cdots\cdots (3, 3, n)$

$(4, 4, 5), (4, 4, 6) \cdots\cdots (4, 4, n)$

$(5, 5, 6) \cdots\cdots (5, 5, n)$

$\vdots$

$(n-1, n-1, n)$

$$1+2+3+\cdots\cdots+(n-1) = \frac{(n-1)n}{2}$$

$\therefore$ $\underline{\dfrac{(n-1)n}{2}\text{通り}}$ ……(答)

(2) X, Y, Z が全て異なるのは $n(n-1)(n-2)$ 通りなので、

X, Y, Z のうち少なくとも2つが等しい場合の数は

$$n^3 - n(n-1)(n-2) = n\{n^2 - (n^2 - 3n + 2)\} = n(3n-2)$$

$\therefore$ $\underline{n(3n-2)\text{ 通り}}$ ……(答)

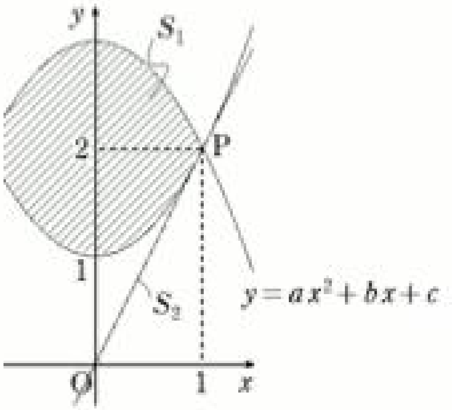
(3) $X < Y < Z$ となるのは ${}_nC_3 = \dfrac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} = \dfrac{n}{6}(n-1)(n-2)$

$\therefore$ $\underline{\dfrac{n}{6}(n-1)(n-2)\text{通り}}$ ……(答)

問 3

(1) $x^2+1=2x \iff (x-1)^2=0 \quad \therefore x=1$

$\therefore \underline{P(1,\ 2)}$ ……(答)



(2) $C_1: y=ax^2+bx+c$ が $P(1,\ 2)$ を通るので，

$2=a+b+c$ ……①

次に，

$$x^2+1=ax^2+bx+c$$

$$\iff x^2+1=ax^2+bx+2-a-b \quad (①より)$$

$$\iff \{(1-a)x-(a+b-1)\}(x-1)=0$$

$$\therefore \underline{\alpha=\frac{a+b-1}{1-a}} \quad ……(答)$$

(3)

$$ax^2+bx+c=2x$$

$$\iff ax^2+(b-2)x+2-a-b+2 \quad (①より)$$

$$\iff \{ax-(2-a-b)\}(x-1)=0$$

$$\therefore \underline{\beta=\frac{2-a-b}{a}} \quad ……(答)$$

(4)

$$\frac{S_1}{S_2}=\frac{\left|\int_\alpha^1 (a-1)(x-\alpha)(x-1)dx\right|}{\left|\int_\beta^1 a(x-\beta)(x-1)dx\right|}$$

$$=\frac{\left|\frac{a-1}{6}(1-\alpha)^3\right|}{\left|\frac{a}{6}(1-\beta)^3\right|}$$

$$=\frac{(1-a)\left(\frac{2-2a-b}{1-a}\right)^3}{-a\left(\frac{2-2a-b}{-a}\right)^3}$$

$$=\frac{(-a)^2}{(1-a)^2}$$

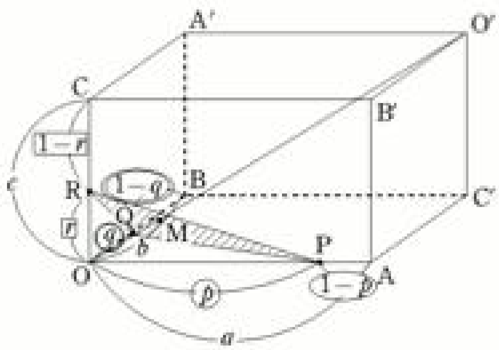
$$=\frac{1}{2}$$

より，$a^2+2a-1=0$

$\therefore \underline{a=-1-\sqrt{2}\ (<0)}$ ……(答)

問４

(1) $\mathrm{OO}'=\underline{\sqrt{a^2+b^2+c^2}}$ ……(答)



(2) $\overrightarrow{\mathrm{OM}}=k\left(\overrightarrow{\mathrm{OA}}+\overrightarrow{\mathrm{OB}}+\overrightarrow{\mathrm{OC}}\right)$

$=\dfrac{k}{p}\overrightarrow{\mathrm{OP}}+\dfrac{k}{q}\overrightarrow{\mathrm{OQ}}+\dfrac{k}{r}\overrightarrow{\mathrm{OR}}$

$\left(\begin{array}{l} k：実数 \\ \overrightarrow{\mathrm{OA}}=\dfrac{1}{p}\overrightarrow{\mathrm{OP}} \\ \overrightarrow{\mathrm{OB}}=\dfrac{1}{q}\overrightarrow{\mathrm{OQ}} \\ \overrightarrow{\mathrm{OC}}=\dfrac{1}{r}\overrightarrow{\mathrm{OR}} \end{array}\right)$

ここで，点Mは平面PQR上より，

$\left(\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{r}\right)k=1$

$\Longleftrightarrow k=\dfrac{pqr}{pq+qr+rp}$

$\left|\overrightarrow{\mathrm{OA}}\right|=a,\ \left|\overrightarrow{\mathrm{OB}}\right|=b,\ \left|\overrightarrow{\mathrm{OC}}\right|=c,\ \overrightarrow{\mathrm{OA}}\cdot\overrightarrow{\mathrm{OB}}=\overrightarrow{\mathrm{OB}}\cdot\overrightarrow{\mathrm{OC}}=\overrightarrow{\mathrm{OC}}\cdot\overrightarrow{\mathrm{OA}}=0$ に注意して

$\left|\overrightarrow{\mathrm{OM}}\right|^2=\left(\dfrac{pqr}{pq+qr+rp}\right)^2\left(\left|\overrightarrow{\mathrm{OA}}\right|^2+\left|\overrightarrow{\mathrm{OB}}\right|^2+\left|\overrightarrow{\mathrm{OC}}\right|^2+2\overrightarrow{\mathrm{OA}}\cdot\overrightarrow{\mathrm{OB}}+2\overrightarrow{\mathrm{OB}}\cdot\overrightarrow{\mathrm{OC}}+2\overrightarrow{\mathrm{OC}}\cdot\overrightarrow{\mathrm{OA}}\right)$

$=\left(\dfrac{pqr}{pq+qr+rp}\right)^2(a^2+b^2+c^2)$

$\therefore\ \underline{\mathrm{OM}=\left|\overrightarrow{\mathrm{OM}}\right|=\dfrac{pqr}{pq+qr+rp}\sqrt{a^2+b^2+c^2}}$ ……(答)

(3) M≡G（Gは三角形PQRの重心）より，

$\begin{cases}\dfrac{k}{p}=\dfrac{1}{3}\\ \dfrac{k}{q}=\dfrac{1}{3}\\ \dfrac{k}{r}=\dfrac{1}{3}\end{cases}\Longleftrightarrow\begin{cases}p=3k\\ q=3k\quad(k>0)\\ r=3k\end{cases}$

$\therefore\ p:q:r=3k:3k:3k=\underline{1:1:1}$ ……(答)

(4) 三角形PQRの垂心をHとする．

実数 l, m を用いて，$\overrightarrow{\mathrm{OH}}=(1-l-m)\overrightarrow{\mathrm{OP}}+l\overrightarrow{\mathrm{OQ}}+m\overrightarrow{\mathrm{OR}}$

$\overrightarrow{\mathrm{PH}}\cdot\overrightarrow{\mathrm{QR}}=\left\{(-l-m)\overrightarrow{\mathrm{OP}}+l\overrightarrow{\mathrm{OQ}}+m\overrightarrow{\mathrm{OR}}\right\}\cdot\left(\overrightarrow{\mathrm{OR}}-\overrightarrow{\mathrm{OQ}}\right)$

$=m\left|\overrightarrow{\mathrm{OR}}\right|^2-l\left|\overrightarrow{\mathrm{OQ}}\right|^2$

$=m(cr)^2-l(bq)^2=0$ ……①

$\overrightarrow{\mathrm{QH}}\cdot\overrightarrow{\mathrm{RP}}=\left\{(1-l-m)\overrightarrow{\mathrm{OP}}+(l-1)\overrightarrow{\mathrm{OQ}}+m\overrightarrow{\mathrm{OR}}\right\}\cdot\left(\overrightarrow{\mathrm{OP}}-\overrightarrow{\mathrm{OR}}\right)$

$=(1-l-m)\left|\overrightarrow{\mathrm{OP}}\right|^2-m\left|\overrightarrow{\mathrm{OR}}\right|^2$

$=(1-l-m)(ap)^2-m(cr)^2=0$ ……②

①，②より，$(1-l-m)(ap)^2=l(bq)^2=m(cr)^2$ ……③

ここで，M≡Hより，$\begin{cases}\dfrac{k}{p}=1-l-m\\ \dfrac{k}{q}=l\\ \dfrac{k}{r}=m\end{cases}$ だから，

③ $\Longleftrightarrow\dfrac{k}{p}(ap)^2=\dfrac{k}{q}(bq)^2=\dfrac{k}{r}(cr)^2\quad(k>0)$

$\Longleftrightarrow a^2p=b^2q=c^2r$

$\therefore\ p:q:r=\underline{\dfrac{1}{a^2}:\dfrac{1}{b^2}:\dfrac{1}{c^2}}$ ……(答)