1

(1) $x^2=t \quad \therefore \ 2xdx=dt$

x	$0 \to \sqrt{\frac{\pi}{2}}$
t	$0 \to \frac{\pi}{2}$

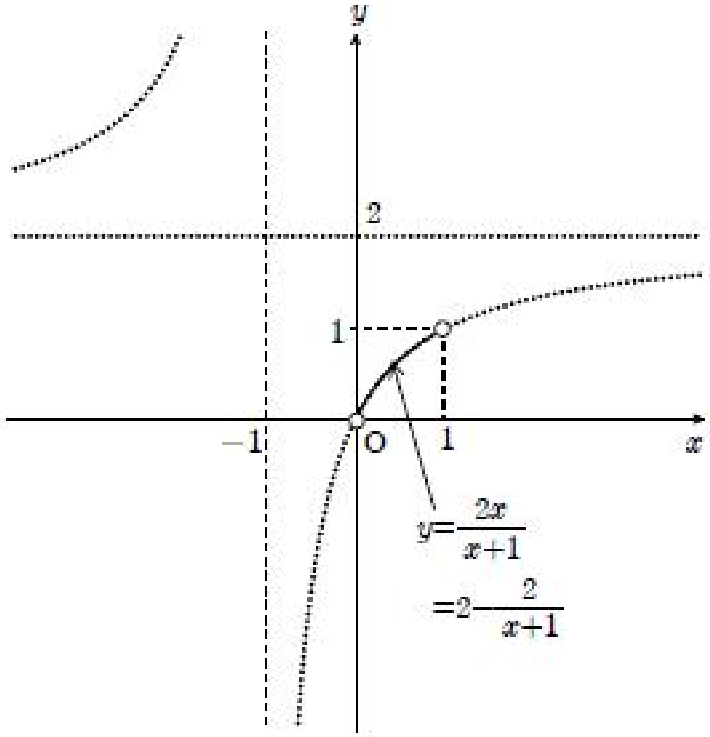
$$\int_0^{\sqrt{\frac{\pi}{2}}} x^3\cos(x^2)dx=\int_0^{\frac{\pi}{2}} t\cos t\cdot\frac{1}{2}dt$$
$$=\frac{1}{2}\int_0^{\frac{\pi}{2}} t(\sin t)'dt$$
$$=\frac{1}{2}\Big[t\sin t\Big]_0^{\frac{\pi}{2}}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin t dt$$
$$=\frac{\pi}{4}-\frac{1}{2}$$ （答）

(2) $\left(\frac{x+1}{2}\right)^{x+1}<x^x \iff (x+1)\log\frac{x+1}{2}<x\log x$

ここで，$f(x)=x\log x-(x+1)\log\frac{x+1}{2}$ $(0<x<1)$ とおく．

$$f'(x)=\log x+1-\log\frac{x+1}{2}-(x+1)\cdot\frac{\frac{1}{2}}{\frac{x+1}{2}}$$
$$=\log\frac{2x}{x+1}$$

x	(0)	$\cdots$	(1)
$f'(x)$		$-$	
$f(x)$		$\searrow$	(0)



増減表より，$f(x)>0$ $(0<x<1)$

つまり，$\left(\frac{x+1}{2}\right)^{x+1}<x^x$ $(0<x<1)$

は示された．（証明終）

2

(1)　$a_n=\left[\dfrac{n}{3}\right]$ $(n=1, 2, 3, \cdots)$ に対して

$$
\begin{aligned}
S_{3n}&=\sum_{k=1}^{3n}(-1)^{k-1}r^{a_k}\\
&=r^{a_1}-r^{a_2}+r^{a_3}-r^{a_4}+\cdots+(-1)^{3n-1}r^{a_{3n}}\\
&=r^0-r^0+r^1-r^1+r^1-r^2+r^2-r^2+r^3-r^3+r^3-r^4+\cdots\cdots\\
&=\underbrace{r^1-r^2+r^3-r^4+r^5\cdots\cdots}_{\substack{\text{初項 } r^1,\text{ 公比 } -r \text{ の等比数列の和}\\(-1<-r<0)}}+\underbrace{(-1)^{3n-1}r^n}_{(0<r<1)}
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \lim_{n\to\infty}S_{3n}&=\lim_{n\to\infty}(r^1-r^2+r^3-r^4+\cdots\cdots)+\lim_{n\to\infty}\underbrace{(-1)^{3n-1}r^n}_{(\to 0)}\\
&=\frac{r^1}{1-(-r)}\\
&=\frac{r}{1+r}\qquad\qquad(\text{答})
\end{aligned}
$$

(2)

$$
\begin{aligned}
T_{2n}&=\sum_{k=1}^{2n}(-1)^{k-1}r^{\frac{b_k}{n}}\\
&=r^{\frac{b_1}{n}}-r^{\frac{b_2}{n}}+r^{\frac{b_3}{n}}-r^{\frac{b_4}{n}}\cdots\cdots+r^{\frac{b_{2n-1}}{n}}-r^{\frac{b_{2n}}{n}}\\
&=\left(r^{\frac{b_1}{n}}+r^{\frac{b_3}{n}}+r^{\frac{b_5}{n}}+\cdots\cdots+r^{\frac{b_{2n-1}}{n}}\right)-\left(r^{\frac{b_2}{n}}+r^{\frac{b_4}{n}}+r^{\frac{b_6}{n}}+\cdots\cdots+r^{\frac{b_{2n}}{n}}\right)\\
&=\left(r^{\frac{1}{n}}+r^{\frac{3}{n}}+r^{\frac{5}{n}}+\cdots\cdots+r^{\frac{2n-1}{n}}\right)-\left(r^{\frac{4}{n}}+r^{\frac{8}{n}}+r^{\frac{12}{n}}+\cdots\cdots+r^{\frac{4n}{n}}\right)
\end{aligned}
$$

$$\times r^{\frac{2}{n}}\quad\times r^{\frac{2}{n}}\quad\times r^{\frac{2}{n}}\cdots\cdots\qquad\qquad\times r^{\frac{4}{n}}\quad\times r^{\frac{4}{n}}\quad\times r^{\frac{4}{n}}\cdots\cdots$$

$$=\sum_{k=1}^{n}r^{\frac{2k-1}{n}}-\sum_{k=1}^{n}r^{\frac{4k}{n}}$$

$$
\begin{aligned}
\therefore\ \lim_{n\to\infty}\frac{1}{n}T_{2n}&=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}r^{\frac{2k}{n}-\frac{1}{n}}-\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}r^{\frac{4k}{n}}\\
&=\int_0^1 r^{2x}dx-\int_0^1 r^{4x}dx\\
&=\left[\frac{r^{2x}}{2\log r}\right]_0^1-\left[\frac{r^{4x}}{4\log r}\right]_0^1=\frac{2r^2-1-r^4}{4\log r}\qquad(\text{答})
\end{aligned}
$$

3

(1)　A を通り，$\overrightarrow{OC}$ に平行な直線を l とする．実数 α を用いて，

$$l:(x,y,z)=(-2,1,3)+\alpha(1,3,4)$$

B を通り，$\overrightarrow{OD}$ に平行な直線を m とする．実数 β を用いて，

$$m:(x,y,z)=(s,3,-1)+\beta(1,2,2)$$

ここで，

$$\begin{cases} -2+\alpha=s+\beta & \cdots\cdots① \\ 1+3\alpha=3+2\beta & \cdots\cdots② \\ 3+4\alpha=-1+2\beta & \cdots\cdots③ \end{cases}$$

①，②，③を同時に満たす実数 α, β が存在 $\begin{cases} \alpha=-6 \\ \beta=-10 \end{cases}$

$\therefore\ s=2, \mathrm{P}(-8,-17,-21)$　　（答）

(2)　$\dfrac{|\overrightarrow{PA}|}{|\overrightarrow{OC}|}=\dfrac{|\overrightarrow{PB}|}{|\overrightarrow{OD}|}=\dfrac{|\overrightarrow{AB}|}{|\overrightarrow{CD}|}$ だから，$\begin{cases} \overrightarrow{PA}=(6,18,24), & |\overrightarrow{PA}|=6\sqrt{26} \\ \overrightarrow{PB}=(10,20,20), & |\overrightarrow{PB}|=30 \end{cases}$

$$\begin{cases} \dfrac{6\sqrt{26}}{\sqrt{26}}=\dfrac{30}{3|t|} \iff |t|=\dfrac{5}{3} \\ \dfrac{6\sqrt{26}}{\sqrt{26}}=\dfrac{6}{\sqrt{9t^2-30t+26}} \iff (3t-5)^2=0 \end{cases} \qquad \therefore\ t=\frac{5}{3}\quad（答）$$

(3)　実数 x，y を用いて

$$\begin{aligned} \overrightarrow{OH} &= \overrightarrow{OP}+x\overrightarrow{PA}+y\overrightarrow{PB} \\ &=(-8,-17,-21)+x(6,18,24)+y(10,20,20) \\ &=(6x+10y-8, 18x+20y-17, 24x+20y-21) \end{aligned}$$

$$\therefore\ \overrightarrow{DH}=\left(6x+10y-\frac{29}{3}, 18x+20y-\frac{61}{3}, 24x+20y-\frac{73}{3}\right)$$

ここで，$\begin{cases} \overrightarrow{DH}\perp\overrightarrow{PA} \\ \overrightarrow{DH}\perp\overrightarrow{PB} \end{cases}$ より，$\begin{cases} 26x+25y=28 & \cdots\cdots④ \\ 10x+10y=11 & \cdots\cdots⑤ \end{cases}$

④，⑤から，$x=\dfrac{1}{2}, y=\dfrac{3}{5}$

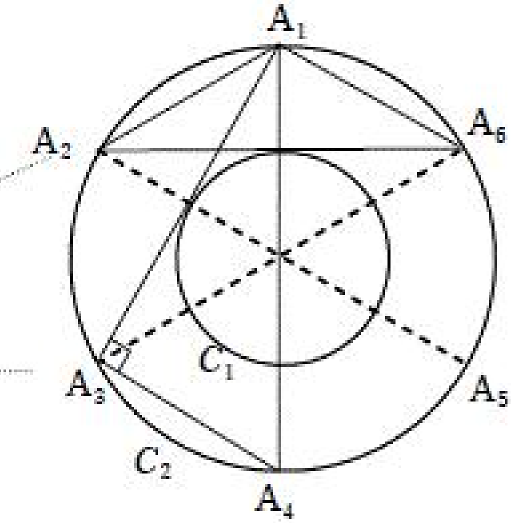
$\overrightarrow{OH}=(1,4,3)$ より，$\mathrm{H}=(1,4,3)$　　（答）

4

(1)

a_0	a_1	a_2	a_3	合計
0	6	12	2	20（個）

（$\triangle A_1A_3A_5$，$\triangle A_2A_4A_6$）



(2)

(a_0) ${}_{n-1}C_2 \cdot 3n = \dfrac{3n}{2}(n-1)(n-2)$

(a_2) $({}_{n+1}C_2 - 1)\cdot 3n = 3n\left\{\dfrac{(n+1)n}{2} - 1\right\}$

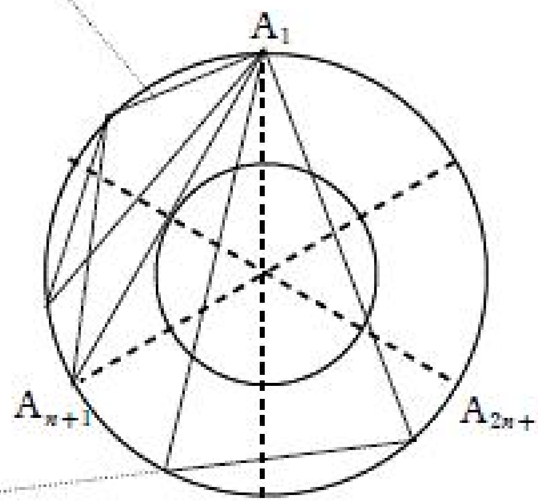
$$= \frac{3n}{2}(n^2+n-2)$$

$$= \frac{3n}{2}(n+2)(n-1)$$

(a_3) n

(a_1) ${}_{3n}C_3 - \dfrac{3n}{2}(n-1)(n-2) - \dfrac{3n}{2}(n^2+n-2) - n$

$$= \frac{3n^2}{2}(n-1)$$

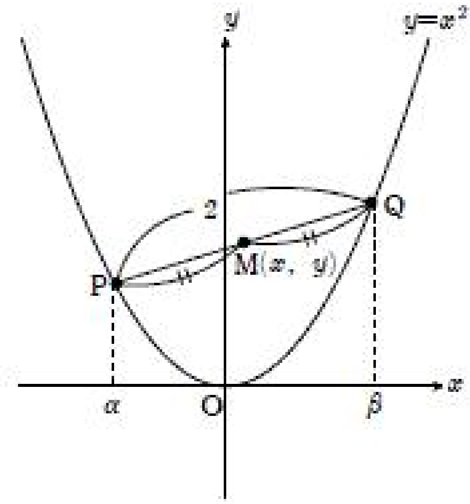


a_0	a_1	a_2	a_3	合計
$\dfrac{3n}{2}(n-1)(n-2)$	$\dfrac{3n^2}{2}(n-1)$	$\dfrac{3n}{2}(n+2)(n-1)$	n	$\dfrac{n}{2}(3n-1)(3n-2)$（個）

（答）

5

(1)　$\mathrm{P}(\alpha, \alpha^2), \mathrm{Q}(\beta, \beta^2)$とする．$(\alpha \neq \beta)$



$$\mathrm{PQ}^2 = (\beta-\alpha)^2 + (\beta^2-\alpha^2)^2 = 4$$

$$\Longleftrightarrow (\beta-\alpha)^2\{1+(\beta+\alpha)^2\} = 4 \quad \cdots\cdots①$$

ここで，2点P，Qの中点を$\mathrm{M}(x, y)$とおく．

$$\begin{cases} x = \dfrac{\alpha+\beta}{2} \\ y = \dfrac{\alpha^2+\beta^2}{2} \end{cases} \Longleftrightarrow \begin{cases} \alpha+\beta = 2x \\ \alpha\beta = 2x^2 - y \end{cases}$$

ここで，α, βを2解にもつtの2次方程式は，

$$t^2 - 2xt + 2x^2 - y = 0 \quad \cdots\cdots②$$

②が異なる2実解にもつので，$D/4 = x^2 - (2x^2 - y) > 0$　　$\therefore\ \underline{\underline{y > x^2}} \quad \cdots\cdots③$

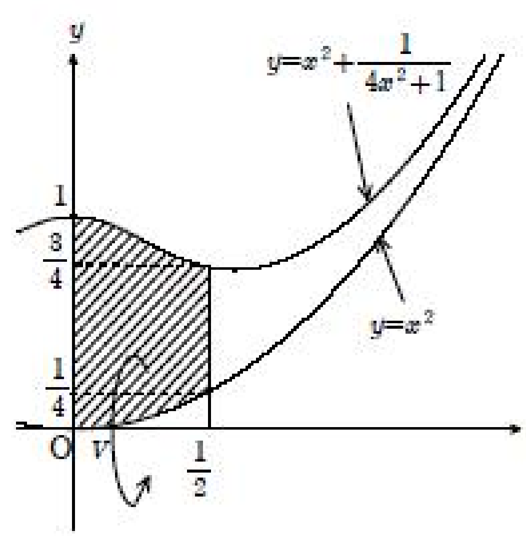
①より，

$$\{(2x)^2 - 4(2x^2-y)\}\{1+(2x)^2\} = 4$$

$$\Longleftrightarrow y = x^2 + \frac{1}{4x^2+1} \quad (\alpha+\beta = 2x \text{ より，} x \text{ は実数全体をとり得る})$$

$\therefore$（Dの方程式）$y = x^2 + \dfrac{1}{4x^2+1}$　　　（答）

(2)



$$V = \pi\int_0^{\frac{1}{2}}\left(x^2 + \frac{1}{4x^2+1}\right)^2 dx - \pi\int_0^{\frac{1}{2}}(x^2)^2 dx$$

$$= \pi\int_0^{\frac{1}{2}}\left(\frac{2x^2}{4x^2+1} + \frac{1}{(4x^2+1)^2}\right)dx$$

$$= \pi\int_0^{\frac{1}{2}}\frac{dx}{2} - \frac{\pi}{2}\underbrace{\int_0^{\frac{1}{2}}\frac{dx}{4x^2+1}}_{(\mathrm{A})} + \pi\underbrace{\int_0^{\frac{1}{2}}\frac{dx}{(4x^2+1)^2}}_{(\mathrm{B})}$$

$x = \dfrac{1}{2}\tan\theta$　　$\therefore\ dx = \dfrac{d\theta}{2\cos^2\theta}$

x	$0 \to \frac{1}{2}$
θ	$0 \to \frac{\pi}{4}$

$$(\mathrm{A}) = \int_0^{\frac{\pi}{4}}\frac{1}{\tan^2\theta+1}\cdot\frac{1}{2\cos^2\theta}d\theta = \frac{1}{2}[\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{8}$$

$$(\mathrm{B}) = \int_0^{\frac{\pi}{4}}\frac{1}{(\tan^2\theta+1)^2}\cdot\frac{1}{2\cos^2\theta}\cdot d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}}\cos^2\theta d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{1+\cos 2\theta}{2}d\theta = \frac{\pi}{16} + \frac{1}{8}$$

$$\therefore\ V = \frac{\pi}{4} - \frac{\pi}{2}\cdot\frac{\pi}{8} + \pi\cdot\left(\frac{\pi}{16} + \frac{1}{8}\right)$$

$$= \frac{3}{8}\pi \qquad \text{（答）}$$