1

$$\left.\begin{array}{l}(\text{赤さいころ}) \longrightarrow x \\ (\text{白さいころ}) \longrightarrow y\end{array}\right\}\ \underline{m \leqq (x,\ y) \leqq M}$$

$$(\text{黄さいころ}) \longrightarrow z$$

(1)

$$\left.\begin{array}{l}(m,\ M)=(1,\ 1)\ \cdots\cdots z=1\quad \text{の}1\text{通り}\\(m,\ M)=(2,\ 2)\ \cdots\cdots z=2\quad \text{の}1\text{通り}\\(m,\ M)=(3,\ 3)\ \cdots\cdots z=3\quad \text{の}1\text{通り}\\(m,\ M)=(4,\ 4)\ \cdots\cdots z=4\quad \text{の}1\text{通り}\\(m,\ M)=(5,\ 5)\ \cdots\cdots z=5\quad \text{の}1\text{通り}\\(m,\ M)=(6,\ 6)\ \cdots\cdots z=6\quad \text{の}1\text{通り}\end{array}\right)\ 6\text{通り}$$

また，$(a,\ y)$ の対称性より，

$$\left.\begin{array}{l}(m,\ M)=(1,\ 2)\ \cdots\cdots z=1,\ 2\quad \text{の}4\text{通り}\\(m,\ M)=(2,\ 3)\ \cdots\cdots z=2,\ 3\quad \text{の}4\text{通り}\\(m,\ M)=(3,\ 4)\ \cdots\cdots z=3,\ 4\quad \text{の}4\text{通り}\\(m,\ M)=(4,\ 5)\ \cdots\cdots z=4,\ 5\quad \text{の}4\text{通り}\\(m,\ M)=(5,\ 6)\ \cdots\cdots z=5,\ 6\quad \text{の}4\text{通り}\end{array}\right)\ 20\text{通り}$$

$$\left.\begin{array}{l}(m,\ M)=(1,\ 3)\ \cdots\cdots z=1,\ 2,\ 3\quad \text{の}6\text{通り}\\(m,\ M)=(2,\ 4)\ \cdots\cdots z=2,\ 3,\ 4\quad \text{の}6\text{通り}\\(m,\ M)=(3,\ 5)\ \cdots\cdots z=3,\ 4,\ 5\quad \text{の}6\text{通り}\\(m,\ M)=(4,\ 6)\ \cdots\cdots z=4,\ 5,\ 6\quad \text{の}6\text{通り}\end{array}\right)\ 24\text{通り}$$

$$\left.\begin{array}{l}(m,\ M)=(1,\ 4)\ \cdots\cdots z=1,\ 2,\ 3,\ 4\quad \text{の}8\text{通り}\\(m,\ M)=(2,\ 5)\ \cdots\cdots z=2,\ 3,\ 4,\ 5\quad \text{の}8\text{通り}\\(m,\ M)=(3,\ 6)\ \cdots\cdots z=3,\ 4,\ 5,\ 6\quad \text{の}8\text{通り}\end{array}\right)\ 24\text{通り}$$

$$\left.\begin{array}{l}(m,\ M)=(1,\ 5)\ \cdots\cdots z=1,\ 2,\ 3,\ 4,\ 5\quad \text{の}10\text{通り}\\(m,\ M)=(2,\ 6)\ \cdots\cdots z=2,\ 3,\ 4,\ 5,\ 6\quad \text{の}10\text{通り}\end{array}\right)\ 20\text{通り}$$

$$\left.(m,\ M)=(1,\ 6)\ \cdots\cdots z=1,\ 2,\ 3,\ 4,\ 5,\ 6\quad \text{の}12\text{通り}\right)\ 12\text{通り}$$

$$\therefore P\ (m \leqq z \leqq M) = \frac{6+20+24+24+20+12}{6^3} = \frac{106}{6^3} = \frac{53}{108}\quad (\text{答})$$

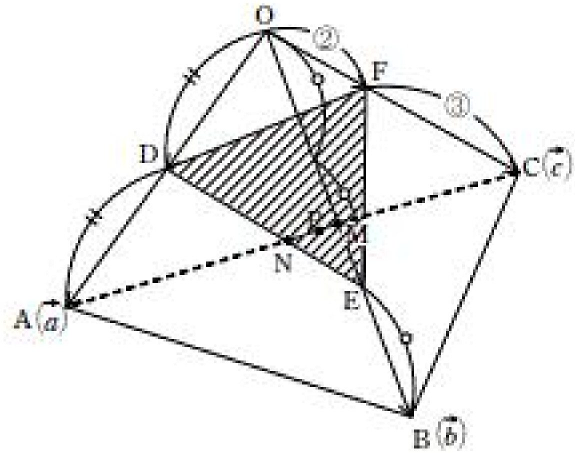
(2)　(1) より，$P\ (m \leqq z \leqq M) = \dfrac{106}{6^3}$ であり，$P\ (m \leqq z \leqq M\ \text{かつ}\ M-m=3) = \dfrac{24}{6^3}$

$$\therefore \underbrace{P_{m \leqq z \leqq M}\,(M-m=3)}_{(\text{条件つき確率})} = \frac{P(m \leqq z \leqq M\ \text{かつ}\ M-m=3)}{P(m \leqq z \leqq M)} = \frac{\frac{24}{6^3}}{\frac{106}{6^3}} = \frac{12}{53}\quad (\text{答})$$

2

(1)

$$\begin{cases} |\vec{a}|=|\vec{b}|=|\vec{c}|=1 \\ \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=\dfrac{1}{2} \end{cases} \quad \cdots\cdots ①$$



$$\begin{aligned}\overrightarrow{\mathrm{MN}}&=\overrightarrow{\mathrm{ON}}-\overrightarrow{\mathrm{OM}}\\ &=\frac{1}{2}(\vec{a}+\vec{c})-\frac{1}{2}\vec{b}\\ &=\frac{1}{2}(\vec{a}+\vec{c}-\vec{b})\end{aligned}$$

ここで,

$$\begin{aligned}|\overrightarrow{\mathrm{MN}}|^2&=\frac{1}{4}\left(1+1+1+2\cdot\frac{1}{2}-2\cdot\frac{1}{2}-2\cdot\frac{1}{2}\right)\ (\because ①)\\ &=\frac{2}{4}\end{aligned}$$

$$\therefore\ |\overrightarrow{\mathrm{MN}}|=\frac{\sqrt{2}}{2} \qquad (答)$$

(2) $\overrightarrow{\mathrm{OP}}$は実数$t$を用いて,

$$\begin{aligned}\overrightarrow{\mathrm{OP}}&=(1-t)\overrightarrow{\mathrm{OM}}+t\overrightarrow{\mathrm{ON}}\\ &=(1-t)\cdot\frac{1}{2}\vec{b}+t\cdot\frac{1}{2}(\vec{a}+\vec{c})\\ &=\frac{t}{2}\overrightarrow{\mathrm{OA}}+\frac{1-t}{2}\overrightarrow{\mathrm{OB}}+\frac{t}{2}\overrightarrow{\mathrm{OC}}\end{aligned}$$

と表せる．さらに，$\overrightarrow{\mathrm{OA}}=2\overrightarrow{\mathrm{OD}}, \overrightarrow{\mathrm{OB}}=\frac{3}{2}\overrightarrow{\mathrm{OE}}, \overrightarrow{\mathrm{OC}}=\frac{5}{2}\overrightarrow{\mathrm{OF}}$を用いると,

$$\overrightarrow{\mathrm{OP}}=t\overrightarrow{\mathrm{OD}}+\frac{3}{4}(1-t)\overrightarrow{\mathrm{OE}}+\frac{5t}{4}\overrightarrow{\mathrm{OF}}$$

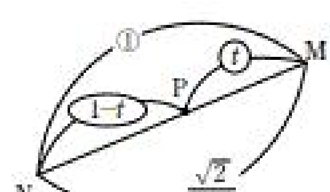
点Pは平面DEF上にあるので,

$$t+\frac{3}{4}(1-t)+\frac{5t}{4}=1 \qquad \therefore\ t=\frac{1}{6}$$

$$\therefore\ \overrightarrow{\mathrm{OP}}=\frac{1}{12}\vec{a}+\frac{5}{12}\vec{b}+\frac{1}{12}\vec{c} \qquad (答)$$

(3) (1), (2)より,

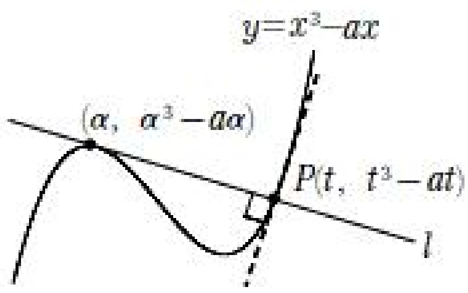
$$\begin{aligned}|\overrightarrow{\mathrm{MP}}|&=\frac{\sqrt{2}}{2}\cdot t\\ &=\frac{\sqrt{2}}{2}\cdot\frac{1}{6} \quad \left(t=\frac{1}{6}\right)\\ &=\frac{\sqrt{2}}{12} \qquad (答)\end{aligned}$$



3

(1) 点$P(t,\ t^3-at)(x>0)$での法線の方程式lは，$y'=3x^2-a$ より，　$a>0$ は必要.

l: $y=-\dfrac{1}{3t^2-a}(x-t)+t^3-at$　（ただし，$t\neq\sqrt{\dfrac{a}{3}}$）



ここで，l との接点を$(\alpha,\alpha^3-a\alpha)$とすると，

$$x^3-ax=-\frac{1}{3t^2-a}x+\frac{t}{3t^2-a}+t^3-at$$

$$\iff\quad x^3+\left(\frac{1}{3t^2-a}-a\right)x+at-t^3-\frac{t}{3t^2-a}=0\qquad\cdots\cdots①$$

$$\iff\quad (x-\alpha)^2(x-t)=0\ \left(\iff x^3-(2\alpha+t)x^2+(\alpha^2+2\alpha t)x-\alpha^2t=0\cdots\cdots②\right)$$

①，②より，

$$\begin{cases}2\alpha+t=0\ \left(\iff\alpha=-\dfrac{t}{2}\right)\\ \alpha^2+2\alpha t=\dfrac{1}{3t^2-a}-a\\ -\alpha^2t=at-t^3-\dfrac{t}{3t^2-a}\end{cases}$$

よって，

$$-\frac{3}{4}t^2=\frac{1}{3t^2-a}-a$$

$$\iff\quad 9t^4-15at^2+4(a^2+1)=0$$

ここで，$t^2=X\ (>0)$ とおくと，$9X^2-15aX+4(a^2+1)=0$　$\cdots\cdots③$

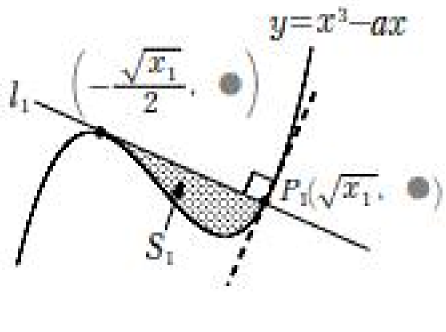
③が異なる2つの正の実数解をもてばいいので，

$$\begin{cases}D=225a^2-4\cdot9\cdot4(a^2+1)>0\\ \dfrac{15}{18}a>0\\ 4(a^2+1)>0\end{cases}\iff\begin{cases}(9a+12)(9a-12)>0\\ a>0\end{cases}$$

$\therefore\ a>\dfrac{4}{3}$　（答）

(2)　$a>\dfrac{4}{3}$ のもとで，$9X^2-15aX+4(a^2+1)=0$ の異なる2つの正の実数解を $X=x_1,x_2\ (=t^2)$

つまり，$t>0$ なので，$t=\sqrt{x_1}$，$\sqrt{x_2}$ より，



$$S_1=-\int_{-\frac{\sqrt{x_1}}{2}}^{\sqrt{x_1}}\left(x+\frac{\sqrt{x_1}}{2}\right)^2\left(x-\sqrt{x_1}\right)dx$$

$$=\frac{1}{12}\left(\sqrt{x_1}+\frac{\sqrt{x_1}}{2}\right)^4$$

$$=\frac{27}{64}x_1^{\,2}$$

同様にして，

$S_2=\dfrac{27}{64}x_2^{\,2}$ より，

$$\begin{aligned}S_1+S_2&=\frac{27}{64}(x_1^{\,2}+x_2^{\,2})\\&=\frac{27}{64}\{(x_1+x_2)^2-2x_1x_2\}\\&=\frac{27}{64}\left\{\left(\frac{5}{3}a\right)^2-2\cdot\frac{4}{9}(a^2+1)\right\}\\&=\frac{3}{64}(17a^2-8)\ （答）\end{aligned}$$

$$\begin{cases}x_1+x_2=\dfrac{5}{3}a\\ x_1\cdot x_2=\dfrac{4}{9}(a^2+1)\end{cases}$$